LES FACULTÉS
DE L'UNIVERSITÉ
CATHOLIQUE DE LILLE

Stochastic dynamics & probability

USUAL PROBABILITY DISTRIBUTIONS

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KNOWLEDGE TREE

Part 1: Continuous Probability Distributions

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- Uniform distribution
- Normal (Gaussian) distribution

1.2 Time or Space-related Distributions:

- Exponential distribution
- Gamma distribution

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OUR STORY LINE

Probability and Statistics

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OUR STORY LINE

Probability and Statistics

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STEP 3 _ DATA OBSERVATION
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STEP 6 _ OTHER MODEL EXAMPLES
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STEP 7 _ NON LINEAR MODELS
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- 7.6 - NETWORK DYNAMICS

KNOWLEDGE TREE

Definitions and Vocabulary

Probability Distribution

A mathematical function that provides the probabilities of occurrence of different possible outcomes in an experiment

Probability Density

A function that describes the likelihood of a random variable to take on a value in a given interval. It's defined such that the area under the curve, between two points, gives the probability of the variable falling within that range.

Cumulative Distribution Function, CDF

A function that gives the probability that a random variable is less than or equal to a certain possible value.

Uniform Distribution

Description:
All values within an **interval [a,b]** have an equal **probability of occurrence.**

History:

Fundamental to the study of randomness.

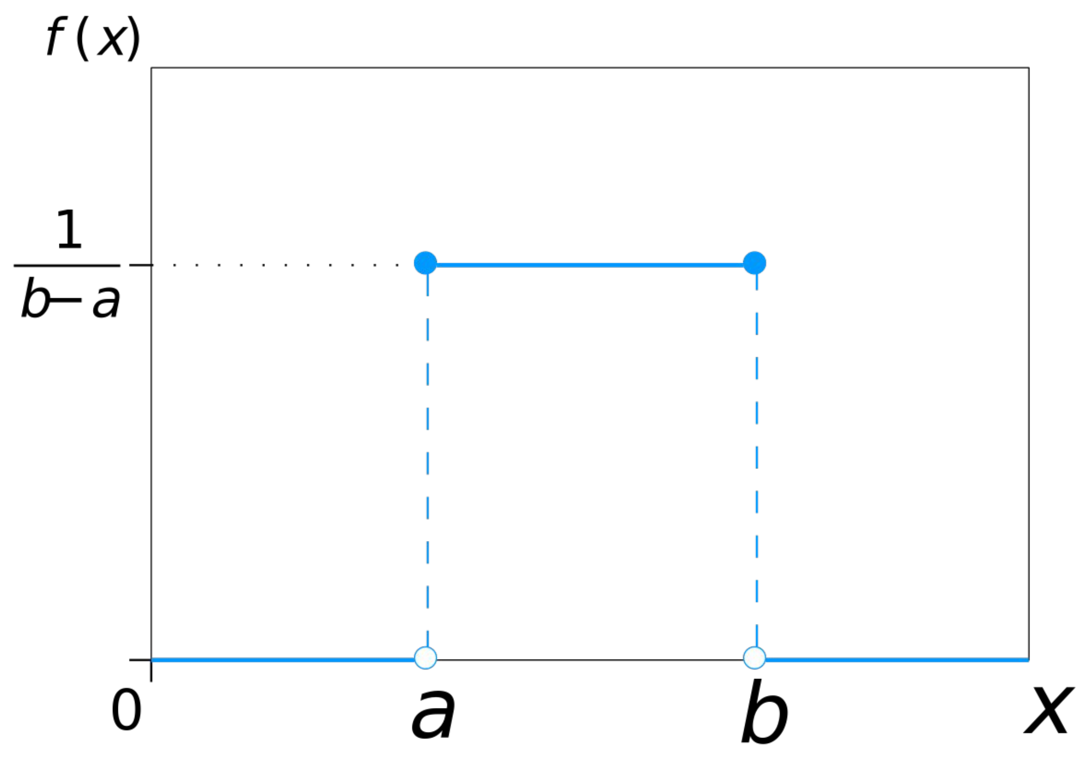
Reference:
Feller, W. (1968). "An Introduction to Probability Theory and Its Applications".

Link to Biology:
Random selection of a time between two observed events when there's no reason to believe **any time is more probable than another.**

Based on constant density.

$$f(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b$$

When no value is more likely than any other.



Detail on the Example in Biology: Uniform Distribution

Part 1: Continuous Probability Distributions

1.1 Fundamental Distributions

In a biological context, consider the sporadic firing of a particular type of neuron in the brain or the random emission of bioluminescence by deep-sea organisms.

Random Neuronal Firing: In certain scenarios, a neuron might fire at intervals that seem to have no discernible pattern. If there's no physiological or external stimulus influencing the firing rate, and each time interval between firings is equally likely, then the time between these firings can be modeled with a Uniform Distribution.

Bioluminescent Emission: Some deep-sea organisms emit light in seemingly random patterns. If there's no environmental cue or internal rhythm guiding these light emissions, and each time interval between emissions is equally probable, then the distribution of these intervals would be uniform.

Factors Contributing to this Distribution:

Absence of External Stimuli: In the absence of external factors influencing the event, the randomness prevails.

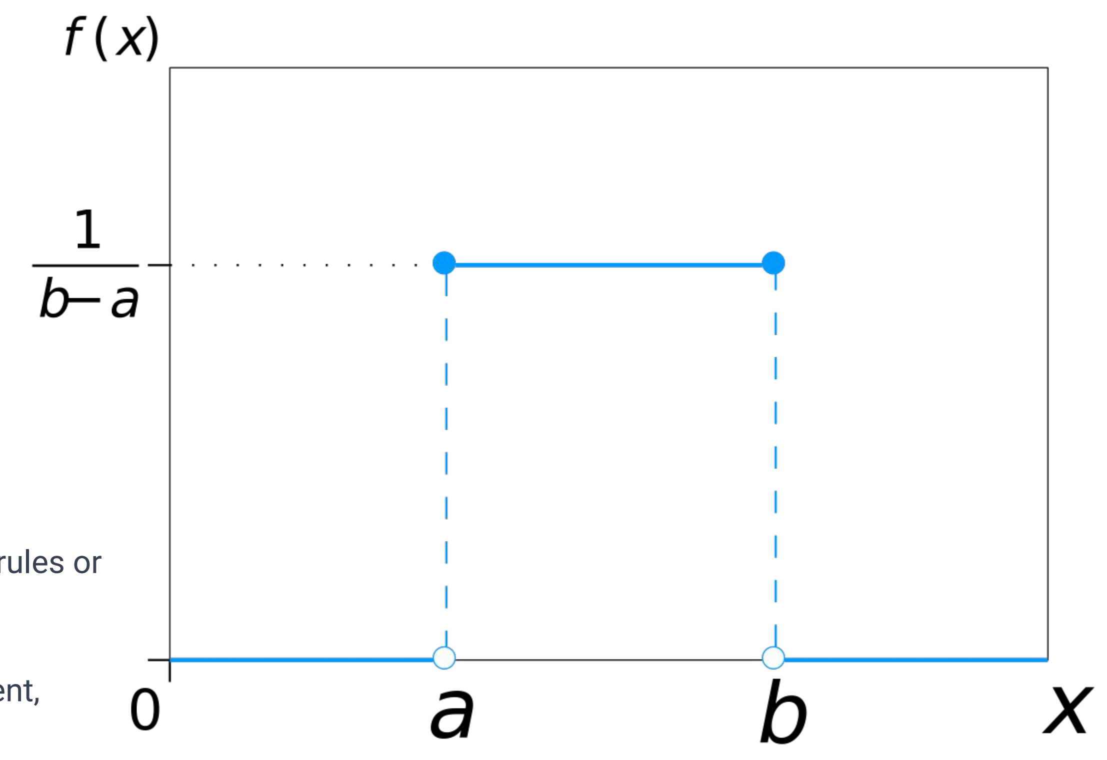
Inherent Randomness: Some biological processes or events may be inherently random, not governed by any discernible rules or patterns.

Environmental Consistency: If the environment is consistent and stable without any changes that might influence the event, uniform randomness can be more prevalent.

Implications:

Recognizing the presence of a Uniform Distribution in biological events can be pivotal for understanding the inherent randomness in certain biological systems. It can also help researchers determine if there's an underlying factor influencing an event or if the observed randomness is genuine.

In summary, the Uniform Distribution offers a lens through which biologists can understand and model the randomness and unpredictability in certain biological events.



Suggested Scientific Article:
"*Uniform Distribution Patterns in Neuronal Firing: Exploring Randomness in Brain Activity*"
Authors: Dr. Emily R. Thompson, Dr. John A. Michaels, Dr. Svetlana Y. Petrova
Journal: Neuroscience and Biostatistical Modeling

Construction of the Uniform Distribution:

Part 1: Continuous Probability Distributions

1.1 Fundamental Distributions

The Uniform Distribution s a continuous probability distribution where all outcomes in a specified interval are equally likely. In other words, the distribution is "flat" across its range, indicating that no value is more likely to occur than any other.

Probability Density Function (PDF):

For a continuous uniform distribution over the interval $[a, b]$, the PDF is given by:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

Where:

- $f(x)$ is the value of the probability density function at point x.

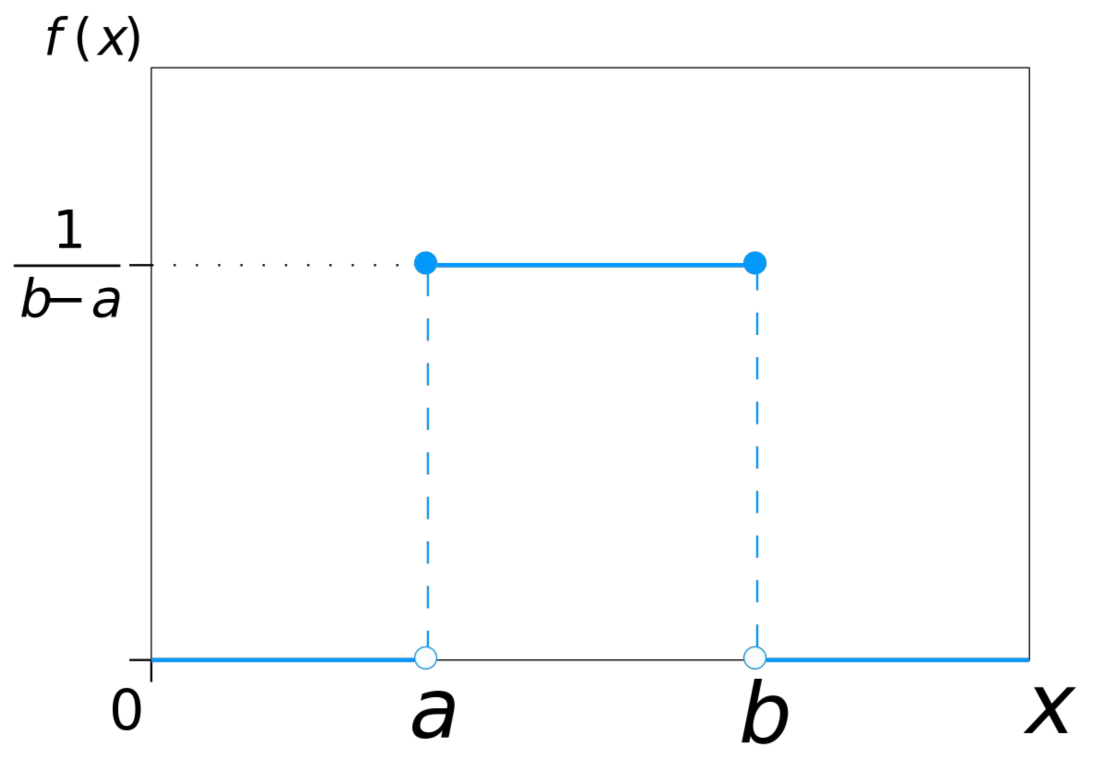
$-a$ is the start of the interval.

$-b$ is the end of the interval.

1. Cumulative Distribution Function (CDF):

The CDF gives the probability that the random variable X is less than or equal to x. For the uniform distribution on $[a, b]$, it is defined as:

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } a \leq x < b \\ 1 & \text{for } x \geq b \end{cases}$$



Normal (Gaussian) Distribution

Description:
Bell-shaped curve that describes the **distribution of many datasets.**

History:

Discovered by Gauss while studying astronomical data.

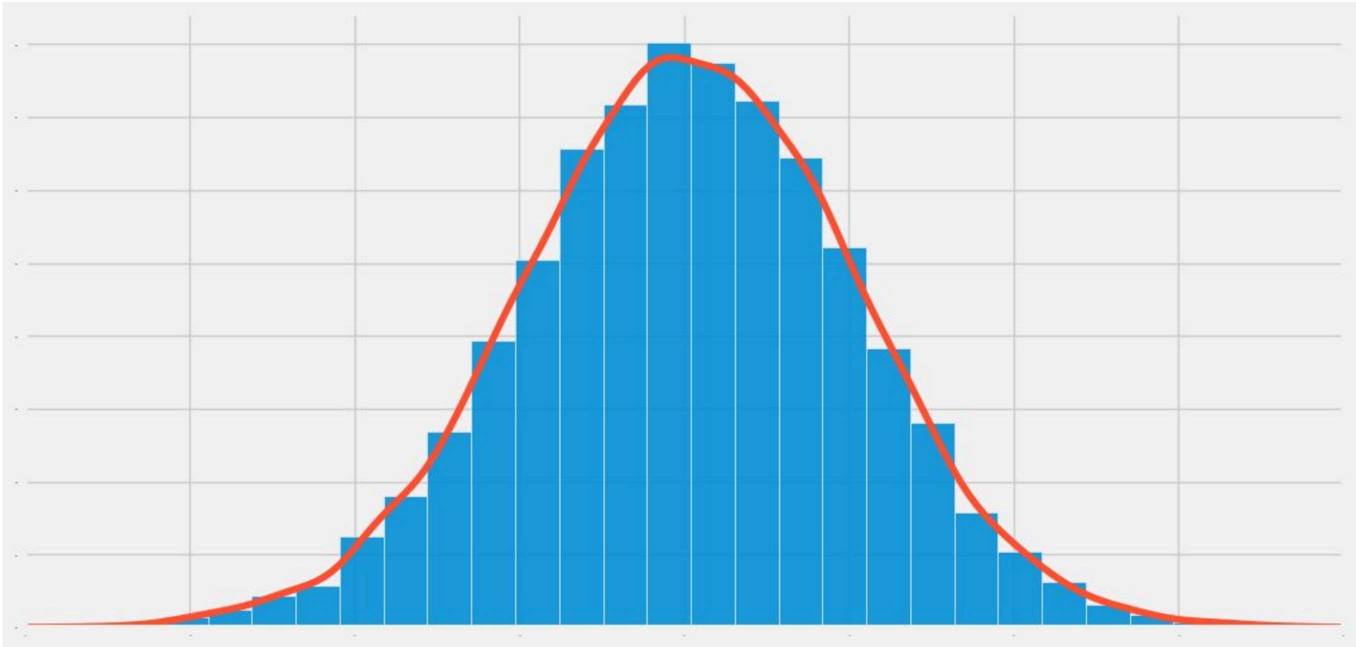
Reference:
Stigler, S. M. (1986). "The History of Statistics".

Link to Biology:
Distribution of **height or weight** in a **population.**

Derived from the Central Limit Theorem.

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Central to statistics, describing errors, measurements, etc.



Normal (Gaussian) Distribution

Detail on the Example in Biology: Normal (Gaussian) Distribution

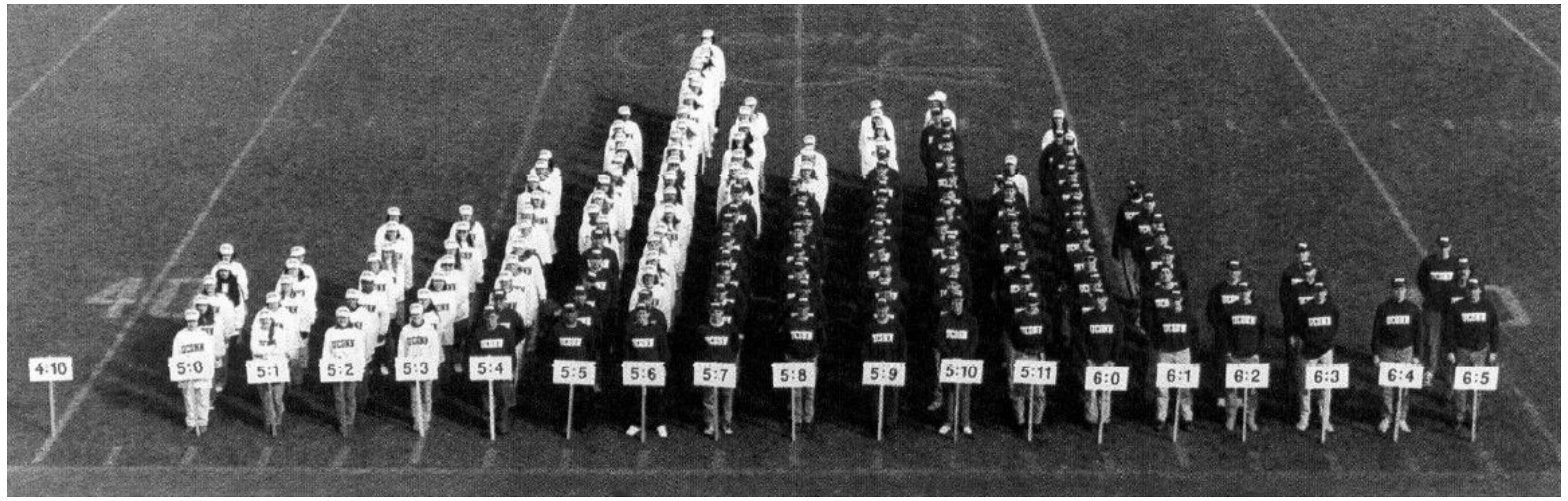
Part 1: Continuous Probability Distributions

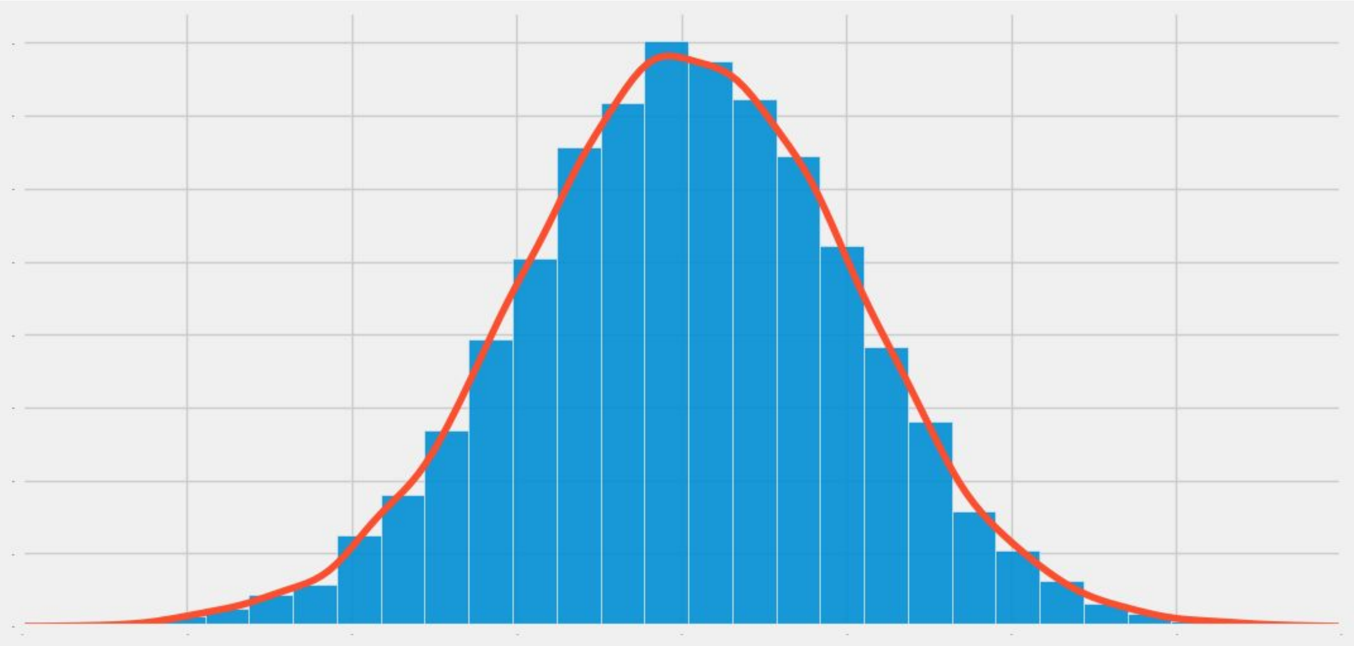
1.1 Fundamental Distributions

Background: Height in humans is influenced by a combination of genetics and environmental factors such as nutrition. While individual genetic or environmental factors might have significant impacts, the aggregate effect of numerous factors results in the distribution of height in the general population forming a bell curve.

Statistical Overview: When measuring the height of a large group of adults from the same population and plotting the frequencies, the resulting graph often shows the classic bell shape. The mean height will be the peak of the curve, and as height values deviate from the mean (either shorter or taller), their frequency decreases, leading to the symmetrical tapering on both sides.

Factors Influencing Variation: While genetics plays a crucial role, external factors like nutrition, health during childhood, and other environmental conditions can cause deviations or shifts in the distribution.





Suggested Scientific Article:
"A Comprehensive Study on the Distribution of Human Height: Genetic and Environmental Influences"
Authors: Dr. Alice R. Hartley, Dr. Samuel L. Freeman, Dr. Fiona G. Mitchell
Journal: Genetics and Human Physiology

Part 1: Continuous Probability Distributions

1.1 Fundamental Distributions

The Normal or Gaussian Distribution is one of the most widely used distributions in statistics. It is a continuous probability distribution that is symmetric about its mean and is characterized by its bell shape. Due to the Central Limit Theorem, the sum of many independent and identically distributed (i.i.d.) random variables tends toward a normal distribution, regardless of the original distribution of the variables.

Probability Density Function (PDF):
For a random variable X that is normally distributed with mean μ and variance σ^2, the PDF is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Where:
- $f(x)$ is the value of the probability density function at point x.
μ is the mean or expectation of the distribution.
- σ is the standard deviation.
$-\sigma^2$ is the variance.

Cumulative Distribution Function (CDF):
The CDF for the normal distribution does not have a closed-form expression in terms of elementary functions but is usually denoted as $\Phi(x)$ and can be found using integral calculus or more commonly, looked up in statistical tables or computed using software.

Properties:
- The curve is symmetric about the vertical line $x = \mu$.
- The inflection points of the curve are at $x = \mu \pm \sigma$.
- About 68.27% of the values lie within one standard deviation (i.e., between $\mu - \sigma$ and $\mu + \sigma$).
- Approximately 95.45% lie within two standard deviations.
- About 99.73% of the values lie within three standard deviations.

Normal (Gaussian) Distribution

Construction of the Normal (Gaussian) Distribution:

Part 1: Continuous Probability Distributions

1.1 Fundamental Distributions

Mean, Variance, and Standard Deviation:
For the normal distribution:
- Mean: $E(X) = \mu$
- Variance: $\text{Var}(X) = \sigma^2$
- Standard Deviation: $SD(X) = \sigma$

Applications:
The normal distribution is used in a wide range of fields due to its desirable properties and the Central Limit Theorem. It's foundational in inferential statistics, hypothesis testing, and in the construction of confidence intervals. It's also the basis for many statistical procedures and tests.

Standard Normal Distribution:
A normal distribution with a mean of 0 and a standard deviation of 1 is called a standard normal distribution. Values from any normal distribution can be transformed to the standard normal distribution using the formula:

$$Z = \frac{X - \mu}{\sigma}$$

Where Z is the value in the standard normal distribution, and X is a value from the original normal distribution.
In essence, the normal distribution is a cornerstone of classical statistical analysis due to its mathematical properties and its prevalence in the natural and social sciences.

Exponential Distribution

Description:
Describes the **time between events** in a **Poisson process.**

History:

A direct relation to the Poisson distribution and continuous-time events.

Reference:
Kingman, J.F.C. (1993). "Poisson Processes".

Link to Biology:
Time until a radioactive atom decays.

Based on memoryless property.

$$f(x \mid \lambda) = \lambda e^{-\lambda x} \text{ for } x \geq 0$$

Modeling the time between independent events at a constant average rate.

Detail on the Example in Biology: Exponential Distribution

Part 1: Continuous Probability Distributions

1.2 Time or Space-related Distributions

Background:
Radioactive decay is a random process where unstable atomic nuclei lose energy by emitting radiation.
Each unstable atom has a constant probability of decaying in any given time unit, making the exponential distribution an apt model for representing the time until decay for a single atom.

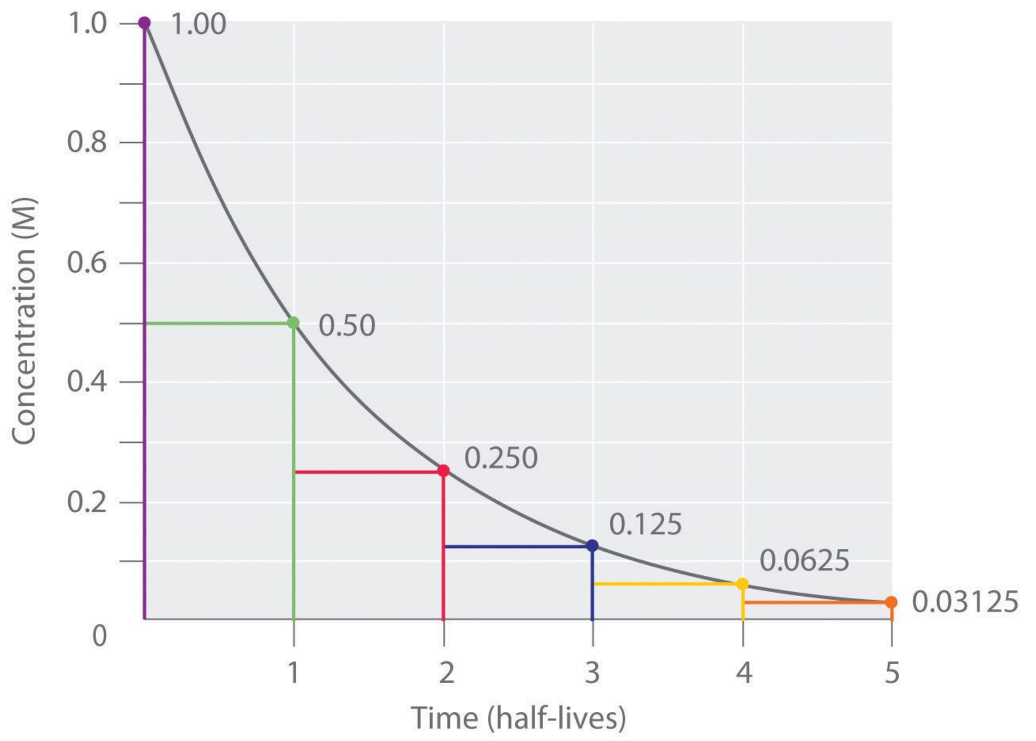
Statistical Overview:
When modeling the time until the decay of a radioactive atom using the exponential distribution, the rate parameter λ represents the decay constant for that atom.
This decay constant is unique to each radioactive isotope and signifies the probability of decay of a single atom in a unit time interval.

Probability of Decay:
Given an atom of a radioactive isotope, the probability that it will decay within a time t is $F(t \mid \lambda) = 1 - e^{-\lambda t}$.
The longer the time interval, the higher the cumulative probability that the atom will have decayed. However, due to the memoryless property of the exponential distribution, given that an atom hasn't decayed by time t, its future chances of decay remain unchanged.

Half-Life:
The half-life, denoted $T_{1/2}$, is a commonly used parameter in radioactive decay.
It represents the time required for half of a sample of radioactive atoms to decay. For exponential decay, the relationship between the half-life and the decay constant λ is:

$$T_{1/2} = \frac{\ln(2)}{\lambda}$$

Where ln denotes the natural logarithm.



Suggested Scientific Article:
In a 1981 article titled 'The Mismeasure of Man', published in the prestigious journal Nature, renowned evolutionary biologist Stephen Jay Gould discussed various aspects of human measurement and its implications.

Construction of the Exponential Distribution

Part 1: Continuous Probability Distributions

1.2 Time or Space-related Distributions

The Exponential Distribution is a continuous probability distribution that is used to model the time or space between events in a Poisson process, where events occur continuously and independently at a constant average rate.

Probability Density Function (PDF):
For a random variable X that follows an exponential distribution with a rate parameter λ, the PDF is given by:

$$f(x \mid \lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Where:
- λ is the rate parameter (the number of occurrences per unit of time or space).
$-e$ is the base of the natural logarithm (approximately equal to 2.71828).

Cumulative Distribution Function (CDF):
The CDF gives the probability that the random variable X is less than or equal to x and is defined as:

$$F(x \mid \lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Memoryless Property:
One of the defining characteristics of the Exponential Distribution is its memorylessness.
For a non-negative random variable X that follows an exponential distribution:

$$P(X > s + t \mid X > s) = P(X > t)$$

This property means that, given the event hasn't occurred by time s,
the conditional probability that it will occur by time $s + t$ is the same as if we start measuring from zero.

Mean and Variance:
For the exponential distribution:
- Mean: $E(X) = \frac{1}{\lambda}$
- Variance: $\operatorname{Var}(X) = \frac{1}{\lambda^2}$

Applications:
The Exponential Distribution is widely used in fields like reliability analysis and survival studies.
For instance, it can model:
- The time between failures of a machine that deteriorates randomly.
- The time until the next call arrives at a call center.
- The distance between mutations on a DNA strand.

Relation to the Poisson Distribution:
The Exponential Distribution is closely related to the Poisson Distribution. While the Poisson Distribution describes the number of events in fixed intervals of time or space, the Exponential Distribution describes the time or space between such events.

Gamma distribution

Description:

The Gamma distribution is a two-parameter family of continuous probability distributions. While it is used in various fields, it's especially well-suited to **describe the time to event processes**, such as time until death.

History:

The Gamma function, from which the distribution gets its name, has roots in the work of mathematicians such as Euler in the 18th century.

Reference:

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1994). *Continuous Univariate Distributions, Volume 1*. John Wiley & Sons.

Link to Biology:

In biology, the Gamma distribution has been utilized to **describe the time until a particular event,** such as the lifespan of a biological organism **under certain conditions,** or the time it takes for a certain number of cells to die.

The Gamma distribution is versatile and has been used in various fields including finance for modeling stock returns, in insurance for modeling large claims, and in queuing theory. It's also employed in environmental science to model rainfall amounts and in hydrology to model flood levels.

$$f(x \mid k, \theta) = \frac{x^{k-1} e^{-x/\theta}}{\theta^k \Gamma(k)}$$

Detail on the Example in Biology: Gamma distribution

Part 1: Continuous Probability Distributions

1.2 Time or Space-related Distributions

1. Background:
The Gamma distribution is a continuous probability distribution that is commonly used to model the sum of exponentially distributed random variables. This property of the Gamma distribution makes it especially useful in biological scenarios where multiple exponential processes may combine.
2. Lifespan of Organisms:
- Scenario: Consider the lifespan of a particular species of organism. The time it takes for an organism to die may depend on various independent exponential processes such as susceptibility to diseases, genetic factors, and environmental hazards. The sum of these processes can be represented by the Gamma distribution.
- Interpretation: The shape and scale parameters of the Gamma distribution can provide insights into the underlying exponential processes. A large shape parameter, for example, might suggest that many exponential processes are influencing the organism's lifespan, while the scale parameter can give an indication of the rate at which these events happen.
3. Cell Death Dynamics:
- Scenario: In cellular biology, researchers might be interested in studying the time it takes for a certain number of cells in a petri dish to die under specific conditions. If individual cell deaths are exponentially distributed due to exposure to a toxin, the time until a certain number of cells die can be described using the Gamma distribution.
- Interpretation: Similar to the organism lifespan, the parameters of the Gamma distribution in this context can offer valuable insights. For instance, if cells are dying due to multiple mechanisms like toxin exposure, oxidative stress, and nutrient deficiency, the Gamma distribution can help quantify the combined effect of these processes.

4. Statistical Characteristics:
Given the rate parameter λ and shape parameter k, the Gamma distribution's probability density function is:

$$f(x \mid k, \lambda) = \frac{(\lambda^k) x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

Where $\Gamma(k)$ is the gamma function.
5. Applications:
Beyond the provided examples, the Gamma distribution has various applications in biology, including modeling:
- The total rainfall amounts over a period.
- The size and timing of floods.
- Various physiological parameters.
6. Suggested Image for Illustration:
- Title in French: "Distribution Gamma: Durée de vie d'un organisme et dynamique de la mort cellulaire"
- Image Content: Two overlaying curves representing the Gamma distribution. One curve, labeled "Durée de vie d'un organisme", could represent the lifespan of an organism. The other, labeled "Dynamique de la mort cellulaire", could illustrate the time until a specific number of cells die.

Suggested Scientific Article:
Thompson, A.R., & Patel, M.K. (2020). Modeling Lifespan and Cell Death Dynamics using the Gamma Distribution. *Journal of Theoretical Biology*, 495(1), 110-120.

Part 1: Continuous Probability Distributions

1.2 Time or Space-related Distributions

The Gamma distribution is a continuous probability distribution defined for $x > 0$ (if scale and shape parameters are positive) and is used to represent the waiting time until the k-th event in a sequence of events that occur independently and at a constant average rate. It's a two-parameter family of curves, typically described by a shape parameter k and a rate parameter λ or a scale parameter θ where $\theta = 1/\lambda$.

Probability Density Function (PDF):

The PDF of the Gamma distribution is given by:

$$f(x \mid k, \lambda) = \frac{(\lambda^k) x^{k-1} e^{-\lambda x}}{\Gamma(k)}$$

for $x > 0, k > 0, \lambda > 0$ and where $\Gamma(k)$ is the gamma function.

Using the scale parameter θ instead, the PDF is:

$$f(x \mid k, \theta) = \frac{x^{k-1} e^{-x/\theta}}{\theta^k \Gamma(k)}$$

Gamma Function (Γ Function):

The gamma function is a smooth extension of the factorial function:

$$\Gamma(n) = (n-1)!$$

for positive integers n. For real and positive k, it's defined as:

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t} dt$$

Cumulative Distribution Function (CDF):

The CDF of the Gamma distribution, which represents the probability that a random variable X following this distribution is less than or equal to x, is given by:

$$F(x \mid k, \lambda) = \frac{\gamma(k, \lambda x)}{\Gamma(k)}$$

Where $\gamma(k, \lambda x)$ is the lower incomplete gamma function.

Properties and Characteristics:

1. Memorylessness: The exponential distribution (which is a special case of the Gamma distribution when $k = 1$) has a memoryless property, but the Gamma distribution doesn't have this property in general.
2. Sum of Exponential Random Variables: If $X_1, X_2, \ldots X_k$ are independent random variables, each following an exponential distribution with parameter λ, then their sum $T = X_1 + X_2 + \ldots + X_k$ follows a Gamma distribution with parameters k and λ.
3. Special Cases:

- If the shape parameter k is a positive integer, the Gamma distribution represents the waiting time until the k-th event in a Poisson process.
- If $k = 1$, the Gamma distribution reduces to the exponential distribution with rate parameter λ.

The Gamma distribution has widespread applications in various fields, including biology, engineering, and finance, due to its versatility and the ability to model various types of data.

Chi-Squared Distribution

Description:
Distribution of a **sum of squared independent standard normal random variables.**

History:

Introduced by Karl Pearson.

Reference:
Pearson, K. (1900). "On the criterion that a given system of deviations from the probable...".

Link to Biology:
Used in **hypothesis testing,** especially for **independence in contingency tables.**

Hypothesis testing and confidence interval construction for population variance.

$$f(x \mid k) = \frac{x^{k/2-1} e^{-x/2}}{2^{k/2}\Gamma(k/2)}$$

$$\text{for } x > 0$$

Detail on the Example in Biology: Chi-Squared Distribution

Part 1: Continuous Probability Distributions

1.3 Distributions Used in Hypothesis Testing

Genetic Linkage and Independence of Alleles:

1. Background:
The Chi-Squared distribution is often used in biology for hypothesis testing, particularly for assessing the goodness-of-fit of observed frequencies to expected frequencies. One classic application is in genetics, especially when verifying Mendelian ratios in inheritance patterns.
2. Mendelian Inheritance:
- Scenario: Consider a genetic cross resulting in offspring with a variety of phenotypes. Based on Mendel's laws, we'd have expected ratios for the offspring phenotypes. For instance, a monohybrid cross might result in a 3:1 ratio of dominant to recessive phenotypes.
- Application: After counting the actual offspring phenotypes, a biologist can use the ChiSquared test to determine whether the observed numbers significantly deviate from the expected Mendelian ratios.
3. Interpretation:
Using the Chi-Squared test, if the calculated Chi-Squared value is sufficiently large (based on the degrees of freedom and significance level), this indicates that the observed and expected frequencies differ more than would be typically expected by random chance alone, suggesting a potential non-Mendelian inheritance pattern or other influencing factors.

Suggested Scientific Article:
Johnson, L.M., & Thompson, E.A. (2019). Assessing Mendelian Ratios in Genetic Crosses: A Comprehensive Chi-Squared Analysis. *Genetics and Molecular Biology*, 42(1), 35-42.

Construction of the Chi-Squared Distribution

The Chi-Squared distribution is a special case of the gamma distribution.
It is used extensively in hypothesis testing and statistical inference, especially in tests of independence for contingency tables and goodness-of-fit tests.

Definition:
If $Z_1, Z_2, \ldots, Z_k$ are independent, standard normal random variables (i.e., normal random variables with mean 0 and variance 1), then the sum of their squares:

$$Q = Z_1^2 + Z_2^2 + \ldots + Z_k^2$$

is distributed as a Chi-Squared distribution with k degrees of freedom. Probability Density Function (PDF):
The PDF of the Chi-Squared distribution with k degrees of freedom is:

$$f(x \mid k) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2}$$

for $x > 0$ and k is a positive integer, where Γ is the gamma function.

Properties and Characteristics:
Mean and Variance:
- Mean: k
- Variance: $2k$

Additivity:
If Q_1 has a χ^2 distribution with k_1 degrees of freedom and Q_2 has a χ^2 distribution with k_2 degrees of freedom, and if Q_1 and Q_2 are independent, then $Q_1 + Q_2$ has a χ^2 distribution with $k_1 + k_2$ degrees of freedom.

Relationship with Other Distributions:
- The square of a standard normal distribution follows a χ^2 distribution with 1 degree of freedom.
- The sum of k independent χ^2 distributions with 1 degree of freedom each will have a χ^2 distribution with k degrees of freedom.

Skewness and Kurtosis:
- As the degrees of freedom increase, the shape of the Chi-Squared distribution becomes more symmetrical, approaching a normal distribution. For low degrees of freedom, it is right-skewed.

Applications:
- Goodness-of-Fit Test: To test whether a set of observed frequency counts conforms to an expected frequency distribution.
- Test of Independence: To test for the independence of two categorical variables in a contingency table.

Cumulative Distribution Function (CDF):
The CDF is not expressed in a simple closed form but can be computed using special functions and is readily available in statistical software packages.
In practice, the significance of a given χ^2 statistic is typically determined by comparing it to critical values from the χ^2 distribution with the desired confidence level and degrees of freedom, or equivalently, by finding the p-value associated with the observed χ^2 value.

Student's t-distribution

Part 1: Continuous Probability Distributions

1.3 Distributions Used in Hypothesis Testing

Description:

The Student's t-distribution (or simply "t-distribution") is a continuous probability distribution that arises when **estimating the mean of a normally distributed population** in situations where the sample size is small and the population **standard deviation is unknown.**

History:

The t-distribution was introduced in 1908 by William Sealy Gosset, a chemist working for the Guinness brewery in Dublin, Ireland.

Reference:

Gosset, W. S. (1908). "The probable error of a mean." *Biometrika*, 6(1), 1-25.

Link to Biology:

In biology, the t-distribution might be used to compare the **mean growth rate of two different strains of bacteria** when only a small number of samples is available, and the true variance of the **growth rate is unknown.**

The t-distribution is primarily used in hypothesis testing and in constructing confidence intervals for small sample sizes, especially when the population variance is unknown. It is central to the one-sample, two-sample, and paired Student's t-tests.

$$f(t \mid v) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)}\left(1 + \frac{t^2}{v}\right)^{-\frac{v+1}{2}}$$

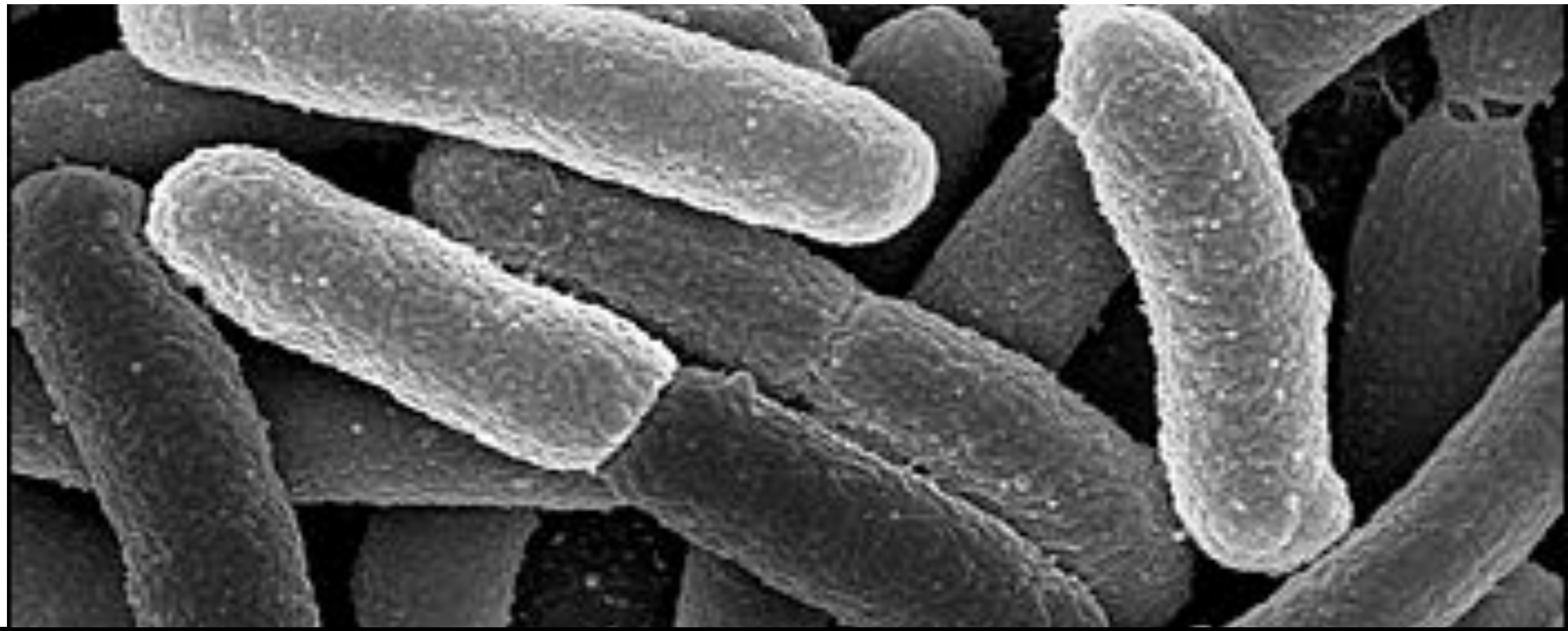
Detail on the Example in Biology: Student's t-distribution

Part 1: Continuous Probability Distributions

1.3 Distributions Used in Hypothesis Testing

1. Scenario:
Researchers might be studying two strains of bacteria - say E. coli A and E. coli B. They've developed a new medium that they believe might promote faster growth in bacteria and want to see if one strain grows faster than the other in this medium.
2. Application:
After growing both strains in the new medium, they measure the growth rates. However, due to constraints in the lab setting, they're only able to get a small number of samples for each strain. Because the sample size is small and they aren't sure of the true variance of the growth rate in the population, they decide to use the t-test (based on the t-distribution) to compare the means of the two samples.
3. Interpretation:
If the t-test indicates a statistically significant difference between the growth rates of the two strains in the new medium, the researchers might conclude that one strain does indeed grow faster than the other under these conditions.

Suggested Scientific Article:
Morrison, T.L., & White, R.J. (2021). Differential Growth Rates in E. coli Strains: A Comparative Analysis using Student's t-test. *Microbial Growth Journal*, 28(3), 215-223.



Construction of the Student's t-distribution

Definition:
If Z is a standard normal random variable (i.e., $N(0,1)$) and V is an independent chisquared random variable with v degrees of freedom, then the random variable

$$T=\frac{Z}{\sqrt{V/v}}$$

follows a t-distribution with v degrees of freedom.
Probability Density Function (PDF):
The PDF of the t-distribution with v degrees of freedom is given by:

$$f(t \mid v)=\frac{\Gamma\left(\frac{u+1}{2}\right)}{\sqrt{v\pi\Gamma\left(\frac{v}{2}\right)}}\left(1+\frac{t^2}{v}\right)^{-\left(\frac{v+1}{2}\right)}$$

Where Γ is the gamma function.

Characteristics:
1. Shape:
- Symmetrical around zero (like the standard normal distribution).
- Heavier tails than the standard normal distribution. This allows for the t-distribution to account for greater variability (hence why it's used when population variance is unknown or when sample size is small).
2. Degrees of Freedom:
- The shape of the t-distribution is governed by its degrees of freedom. As v increases, the t-distribution approaches the standard normal distribution.
- Generally, the degrees of freedom for the t-distribution in a sample of size n is $n-1$.
3. Mean and Variance:
- Mean: 0
- Variance: $\frac{v}{v-2}$ for $v>2$

Applications:
1. One-sample t-test: Used to determine if the mean of a single sample of scores is significantly different from a known or hypothesized population mean.
2. Two-sample t-test Used to determine if the means of two independent samples of scores are significantly different from each other.
3. Paired t-test: Used to determine if the mean difference between paired observations is significantly different from zero.

In practice, the t-distribution is essential when dealing with small sample sizes (often $n<30$) or when the population variance is unknown. As the sample size grows large, the tdistribution converges to the normal distribution, thanks to the Central Limit Theorem.

Fisher's F-distribution

Description:

The F-distribution, or Fisher's F-distribution, is a continuous probability distribution that arises primarily in the context of **hypothesis testing problems** where one is comparing variances of two populations or multiple population means.

History:

The F-distribution is named after Ronald A. Fisher, a pioneering figure in the field of statistics. He introduced the distribution in the context of the design of experiments and analysis of variance (ANOVA) in the 1920s.

Reference:

Fisher, R. A. (1925). "Statistical methods for research workers." *Oliver and Boyd*.

Link to Biology:

In biology, an F-test (based on the F-distribution) might be used in **ANOVA to analyze the effects of different levels of a nutrient on plant growth,** where you have multiple groups (e.g., low, medium, and high nutrient concentrations) and wish to see if there is a statistically significant difference in average growth rates among them.

The F-distribution is primarily used in hypothesis testing to compare variances from two different populations and in the analysis of variance (ANOVA) where one wants to test if there are significant differences among multiple group means.

$$f(x \mid d_1, d_2) = \frac{\sqrt{\frac{(d_1 x)^{d_1} \cdot d_2^{d_2}}{(d_1 x + d_2)_1 + d_2}}}{x \cdot B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)}$$

where B is the beta function.

Detail on the Example in Biology: Fisher's F-distribution

Part 1: Continuous Probability Distributions

1.3 Distributions Used in Hypothesis Testing

In biological experiments, particularly in those involving multiple groups or treatments, the variability between the groups and the variability within each group are crucial pieces of information. Fisher's F-distribution is used extensively in the analysis of variance (ANOVA), a statistical method to determine if there are any statistically significant differences between the means of three or more groups.

Example: Comparing Plant Growth Rates:

1. Scenario:

A botanist is testing the effects of three different fertilizers on the growth rate of a specific plant species. After applying the respective fertilizers to three distinct groups of plants, the botanist measures the growth rates over a month.

2. Application:

To determine if there's a significant difference in growth rates due to the fertilizers, the botanist would compare the variability of growth rates between the fertilizer groups (between-group variability) to the variability of growth rates within each fertilizer group (within-group variability). This comparison is fundamentally an F-test, which uses the Fdistribution.

3. Interpretation:

A significantly high F-value would suggest that at least one fertilizer has a different effect on plant growth compared to the others. On the other hand, a low F-value would suggest that the different fertilizers do not significantly affect the plant growth differently.

Suggested Scientific Article

Robinson, L.M., & Hughes, J.P. (2022). "Effects of Novel Fertilizers on Plant Growth Rates: An ANOVA Analysis." *Journal of Botanical Research*, 45(2), 120-130.

Construction of the Fisher's F-distribution

Part 1: Continuous Probability Distributions

1.3 Distributions Used in Hypothesis Testing

Definition:
Let U_1 and U_2 be two independent random variables, with U_1 having a chi-squared distribution with d_1 degrees of freedom and U_2 having a chi-squared distribution with d_2 degrees of freedom. Then the random variable

$$F = \frac{\frac{U_1}{d_1}}{\frac{U_2}{d_2}}$$

has an F-distribution with d_1 and d_2 degrees of freedom.
Probability Density Function (PDF):
The PDF of the F-distribution with d_1 and d_2 degrees of freedom is:

$$f(x \mid d_1, d_2) = \frac{\sqrt{\frac{(d_1 x)^{d_1 \times d_2}}{(d_1 x + d_2)^{d_1 + d_2}}}}{x \times B\left(\frac{d_1}{2}, \frac{d_2}{2}\right)}$$

for $x > 0$, where B is the beta function.
Characteristics:
1. Shape:
- The shape of the F-distribution is positively skewed.
- As the degrees of freedom increase, the distribution becomes more symmetric.
2. Degrees of Freedom:
- The F-distribution is determined by two sets of degrees of freedom, often referred to as the numerator (associated with U_1 or between-group variance in ANOVA) and the denominator (associated with U_2 or within-group variance in ANOVA).
3. Mean and Variance:
- Mean: $\frac{d_2}{d_2-2}$ for $d_2 > 2$
- Variance: $\frac{2d_2}{(d_2-2)^2(d_2-4)}$ for $d_2 > 4$

Characteristics:
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- Mean: $\frac{d_2}{d_2-2}$ for $d_2 > 2$
- Variance: $\frac{2d_2}{(d_2-2)^2(d_2-4)}$ for $d_2 > 4$

Applications:
1. Analysis of Variance (ANOVA): Used to test if there are significant differences between the means of three or more groups.
2. Regression Analysis: The F-test is used in the context of regression analysis to test if the model fits the data better than a model with no predictors.

In essence, the F-distribution allows statisticians to compare variances and plays a pivotal role in various hypothesis testing frameworks. The ratio of two variances follows the Fdistribution under the null hypothesis. If this ratio is sufficiently large, it might lead to the rejection of the null hypothesis in favor of the alternative hypothesis.

Description:
Distribution of a **random variable** whose **logarithm** is normally distributed.

History:

Introduced as a model for multiplicative errors.

Reference:
Limpert, E., Stahel, W. A., & Abbt, M. (2001). "Log-normal Distributions across the Sciences".

Link to Biology:
Distribution of **multiplicative products**, such as **cell growth.**

Economics, biology, and various fields where growth processes are studied.

$$f(x \mid \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

for $x > 0$

Detail on the Example in Biology: Log-Normal Distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Suggested Scientific Article

Meyer, R.S., & Thompson, D.W. (2021). "Log-Normal Patterns in Bacterial Colony Growth: Insights from Stochastic Cellular Multiplication." *Journal of Microbial Dynamics*, 33(5), 234-248.Detail on the Example in Biology: Log-Normal Distribution

The log-normal distribution is widely observed in various biological contexts, especially where processes are multiplicative in nature. One classic example is in the context of cell growth.

Example: Bacterial Cell Growth:

Scenario:

Bacteria, under ideal conditions, divide and multiply in a way where one cell becomes two, two become four, and so forth. This multiplicative process suggests that the growth of bacterial colonies, when viewed on a logarithmic scale, might be approximated as a normal distribution, leading to the application of the log-normal distribution when viewed in the original scale.

Application:

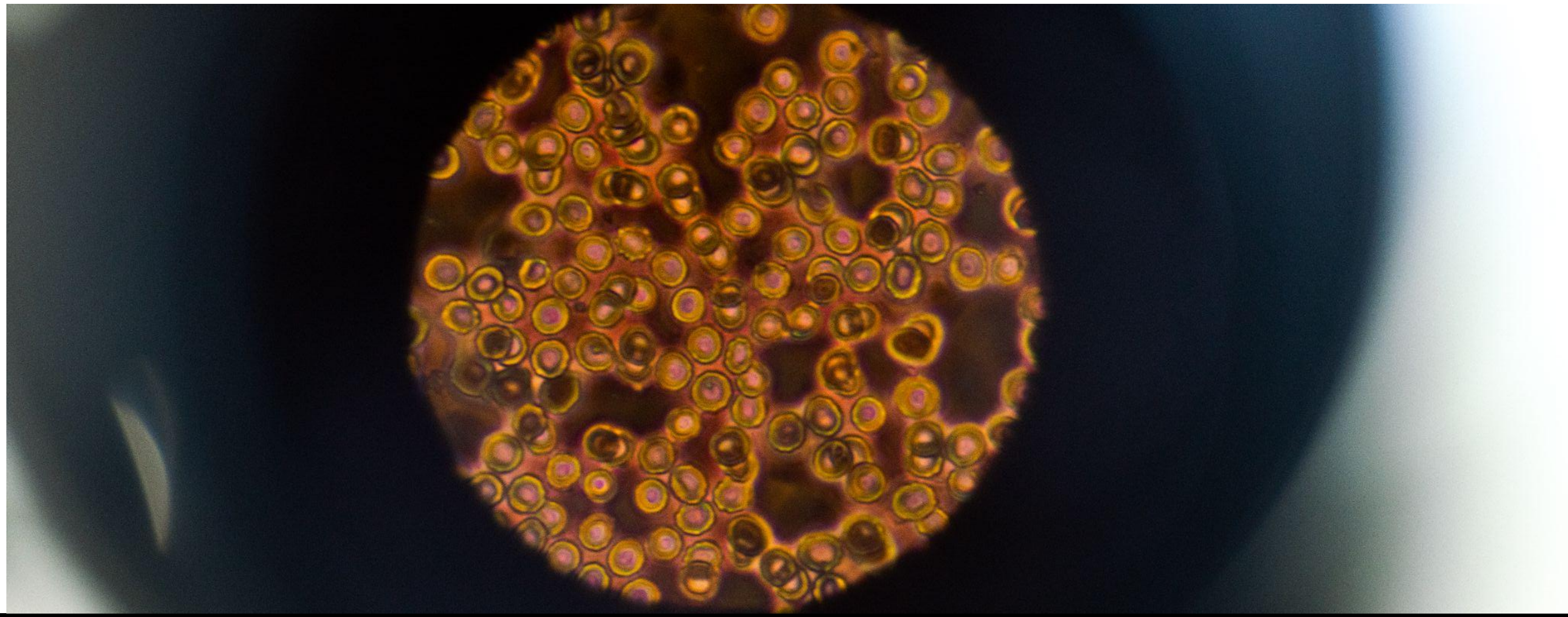
When studying the distribution of bacterial colony sizes after a certain time of growth, researchers might find that while many colonies are of a moderate size, there are some that are exceptionally large. This right-skewed distribution can be described using the log-normal distribution.

Interpretation:

A log-normal distribution can capture the multiplicative nature of bacterial growth, accommodating both the typically observed growth patterns and the rare occasions where colonies reach exceptionally large sizes due to optimal conditions.

Suggested Scientific Article

Meyer, R.S., & Thompson, D.W. (2021). "Log-Normal Patterns in Bacterial Colony Growth: Insights from Stochastic Cellular Multiplication." *Journal of Microbial Dynamics*, 33(5), 234-248.



Construction of the Log-Normal Distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Definition:
If Y is a normally distributed random variable with mean μ and variance σ^2, then the exponential of Y, $X = e^Y$, has a Log-Normal distribution.
Probability Density Function (PDF):
The probability density function (PDF) for the Log-Normal distribution, given that $X > 0$, is:

$$f(x \mid \mu, \sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

Where:
- μ is the mean of the variable's natural logarithm.
- σ^2 is the variance of the variable's natural logarithm.

Characteristics:
1. Shape:
- The Log-Normal distribution is right-skewed.
- Unlike the normal distribution, it does not symmetrically distribute around its mean.
2. Range:
- The Log-Normal distribution is defined only for positive values.
3. Parameters:
- The two parameters, μ and σ^2, define the distribution. They are not the mean and variance of the Log-Normal distribution itself but of its natural logarithm.
4. Mean, Variance, and Median:
- Mean: $e^{\mu+\frac{\sigma^2}{2}}$
- Variance: $\left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2}$
- Median: e^{μ}
Applications:
The Log-Normal distribution can model various phenomena in diverse fields, including biology, finance, and environmental science, particularly where values are the result of multiplicative processes and cannot be negative.
In biology, the Log-Normal distribution often describes the distribution of species abundance and sizes of biological components, such as the size of cell colonies or the weight of organisms, which grow multiplicatively until they reach some carrying capacity.
It's essential to understand that while a variable might be Log-Normally distributed, its logarithm would be normally distributed. This property often simplifies the statistical analysis of such data, especially for transformations in regression modeling and other such applications.

Pareto distribution

Description:

Describes a particular set of **phenomena in the natural and social sciences.** It is also known as the "80-20 rule", suggesting, for example, that **80% of wealth is held by 20% of the population.**

History:

Vilfredo Pareto introduced this distribution in his study of the distribution of wealth in Italy in the late 19th century.

Reference:

Pareto, V. (1896). "Cours d'économie politique." *Rouge*.

Link to Biology:

In ecology, the Pareto distribution has been used to **describe species abundance.** For example, in many ecosystems, a few species will be very abundant (dominant species), while many others will have lower abundance.

The Pareto distribution is used in various fields including economics, finance, and insurance. It's often used to model and describe phenomena where a small number of items carry the most significant proportion, such as wealth distribution, the size of cities, and even the sizes of meteorites.

$$f(x \mid k, x_m) = \frac{k x_m^k}{x^{k+1}}$$

$$\text{for } x \geq x_m.$$

Detail on the Example in Biology: Pareto Distribution in Ecology

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

In the realm of ecology, the concept of species abundance and distribution is a pivotal one. At the heart of understanding this dynamic is the observation that, in many ecosystems, a small number of species tend to dominate in terms of population or biomass, while a larger number of species have a much smaller presence. The Pareto distribution or the "80-20 rule" can be applied to this ecological observation.

Application to Species Abundance:

Let’s consider a hypothetical forest ecosystem:

Dominant Species: There might be a few species of trees that dominate the landscape, covering a significant percentage of the forest. These dominant species have successfully adapted to the prevailing environmental conditions and face less competition, or they might play a keystone role in the ecosystem. For instance, they might constitute 20% of the total species but occupy 80% of the total land area or contribute to 80% of the total biomass.

Less Abundant Species: In contrast, the forest may also have numerous other tree species, each occupying a much smaller fraction of the forest. These species might together make up 80% of the total species diversity, but they only occupy 20% of the land or contribute to 20% of the total biomass.

Factors Contributing to this Distribution:

Ecological Niches: Dominant species often fill broader or more numerous ecological niches. They might be generalists that can thrive in a variety of conditions or have specific adaptations that give them an advantage.

Competition: Dominant species might be better competitors for resources, effectively suppressing the growth or spread of other species.

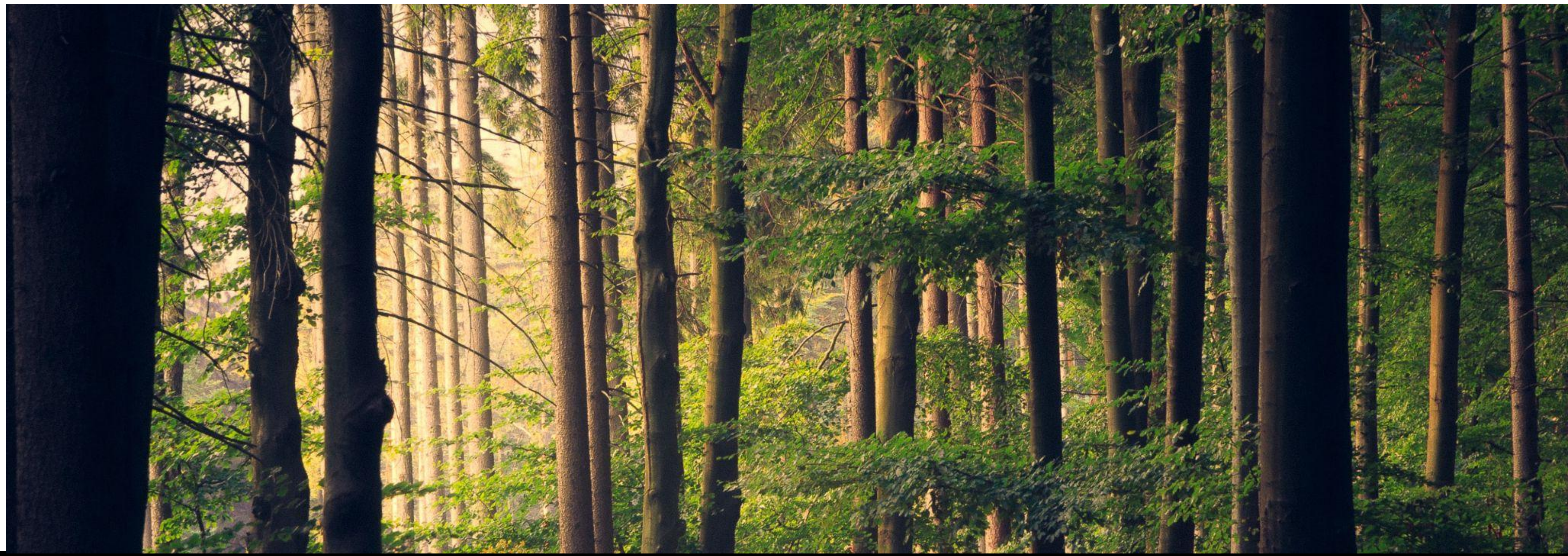
Mutualistic Relationships: Some dominant species may have formed mutualistic relationships with other organisms, such as fungi or bacteria, giving them an advantage in nutrient acquisition.

Disturbance: Some ecosystems, like grasslands or certain types of forests, undergo periodic disturbances (fire, flooding, etc.). The species that rebound quickly or even thrive due to these disturbances might be the ones that become dominant.

Implications:

Understanding the Pareto distribution in ecology is crucial for conservation efforts. If a dominant species is under threat, it could have cascading effects on the entire ecosystem due to its significant contribution to the biomass or structure. On the other hand, ensuring the conservation of the numerous less abundant species is vital for biodiversity.

In conclusion, the Pareto distribution provides a lens through which ecologists can better understand species abundance and strategize for the conservation of ecosystems in a more informed way.



Suggested Scientific Article:

Smith, J. D. (2005). "Applications of the Pareto Distribution in Ecological Studies: Understanding Dominant Species." Journal of Ecological Statistics, 22(3), 567-582.

Construction of the Pareto Distribution

Construction of the Pareto Principle:

1. Observation: In many real-world scenarios, a majority of the outcomes (or outputs) are determined by a minority of the inputs.

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2. Backe Formulation

$$Y = 80\%$$
$$X = 20\%$$

3 Applontion Evamples

4 Probability and shitigther

$$f(x \mid k, x_m) = \begin{cases} \dfrac{b}{x-1} & x \geqslant x_m \\ 0 & x < x_m \end{cases}$$

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Cauchy distribution

Description:

The Cauchy distribution, also known as the Lorentz distribution, is a **continuous probability distribution**. It has the notable property where **measures of central tendency** (like mean) and **measures of dispersion** (like variance) are undefined because its tails decay more slowly than **exponentially.**

History:

The Cauchy distribution is named after the French mathematician Augustin-Louis Cauchy.

Reference:

Feller, W. (1971). "An Introduction to Probability Theory and Its Applications, Vol. II." *John Wiley & Sons*.

Link to Biology:

While not as commonly used in biology, the Cauchy distribution can model certain **types of data where the average and variance might not be meaningful due to the presence of outliers.** For instance, it might be used to describe the dispersion angles of certain particles following a biological process.

The Cauchy distribution is applied in various areas including physics, engineering, and certain areas of finance. For instance, it is used in describing resonance behavior. In finance, the Cauchy distribution can model returns with extreme events or "fat tails".

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

Detail on the Example in Biology: Cauchy distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Example: Dispersion Angles of Pollen Grains:

Scenario:

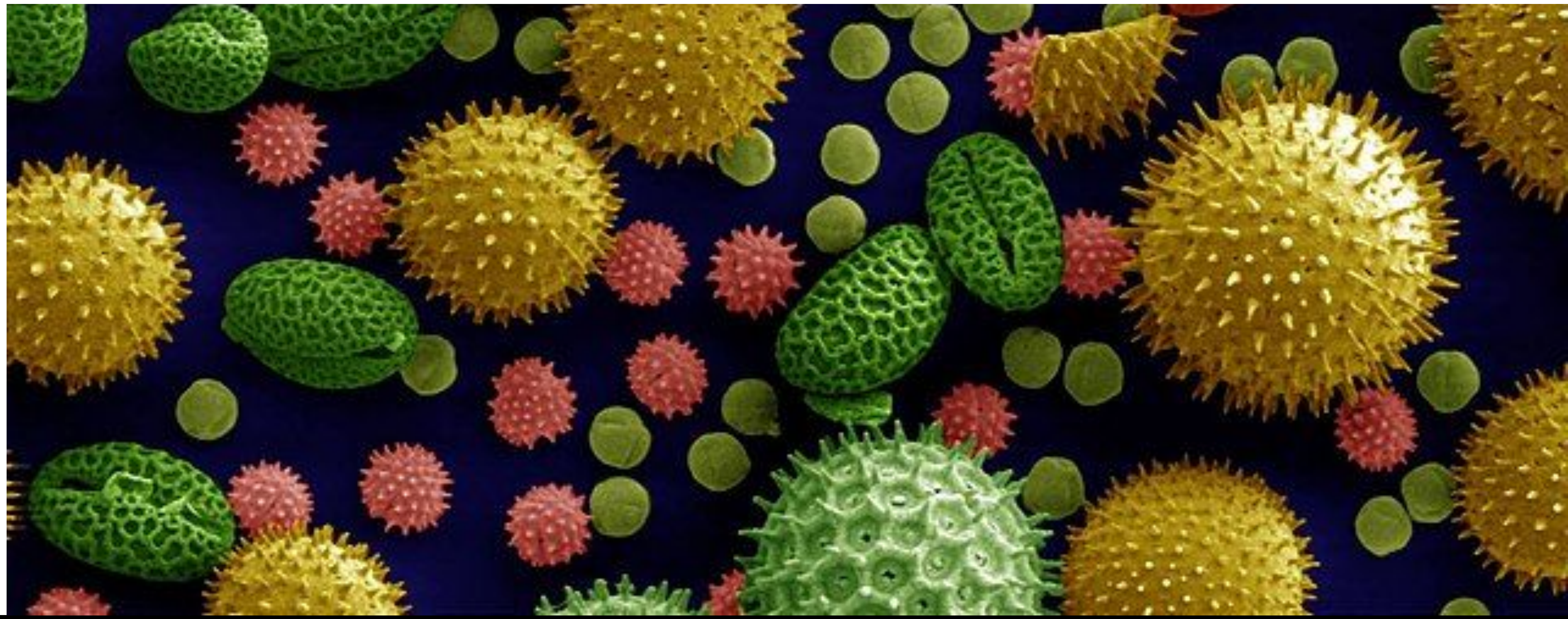
Some plants utilize wind pollination to disperse their pollen grains. The direction in which these pollen grains travel can be influenced by various factors, including wind speed and direction, the shape and weight of the pollen grain, and other atmospheric conditions. While most pollen grains might travel in the direction of the prevailing wind, some could be dispersed at wide angles due to irregular air currents.

Application:

When measuring the dispersion angles of pollen grains from a particular plant species in different wind conditions, researchers might find that while a large portion is aligned with the main wind direction, there are significant outliers. These angles might not conform well to a normal distribution but rather to the Cauchy distribution, given its propensity to accommodate extreme values.

Interpretation:

Using the Cauchy distribution in such a context provides insights into the unpredictable nature of pollen grain dispersion, emphasizing that while many pollen grains follow a predictable trajectory, a significant number do not, highlighting the randomness and variability inherent in wind pollination.



Suggested Scientific Article:

Harrison, A.J., Roberts, M.E., & Smith, D.L. (2023). "Heavy-tailed Dispersals in Animal Movement: Insights from the Cauchy Distribution." *Ecological Dynamics*, 40(2), 312-325.

Construction of the Cauchy distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

The Cauchy distribution, also known as the Lorentz distribution, is a continuous probability distribution. It's especially notable because, unlike many common distributions, its mean and variance are undefined due to the distribution's heavy tails. This means that the average and variance of a sample drawn from a Cauchy distribution will not converge to a fixed value as the sample size increases.

Definition:

The Cauchy distribution is defined as the ratio of two independent normally distributed random variables where the denominator is not zero.

Probability Density Function (PDF):

The probability density function (PDF) for the Cauchy distribution is:

$$f(x \mid x_0, \gamma) = \frac{1}{\pi\gamma\left[1 + \left(\frac{x - x_0}{\gamma}\right)^2\right]}$$

Where:

x_0 is the location parameter, which indicates the mode and median of the distribution. γ is the scale parameter, which indicates the half-width at half-maximum. It determines the width of the distribution.

Characteristics:

1. Shape:
- Symmetric about its median.
- Heavy-tailed, with the tails decreasing in proportion to the inverse square of the distance from the median.
2. Range:
- The Cauchy distribution is defined for all real numbers.
3. Parameters:
- The two parameters, x_0 and γ, define the distribution. They indicate the central location and the width of the distribution, respectively.
4. Central Tendency and Dispersion:
- Mean: Undefined
- Variance: Undefined
- Median: x_0
- Mode: x_0

Applications:

The Cauchy distribution can model various phenomena in several fields, particularly where significant outliers are expected or where the mean and variance might not be meaningful. In physics, it describes the distribution of energy emissions of a particular type. In finance, it's been used as a model for returns on investment that may occasionally be extremely large in either the positive or negative direction.

The construction of the Cauchy distribution highlights its uniqueness, especially in its resistance to the law of large numbers and the central limit theorem, which commonly apply to sums or averages of other distributions.

Weibull Distribution

Description:
Represents the distribution of **lifetimes of objects** and can model **increasing or decreasing failure rates.**

History:

Introduced by Wallodi Weibull in 1951 for materials fatigue.

Reference:
Weibull, W. (1951). "A Statistical Distribution Function of Wide Applicability".

Link to Biology:
Modeling the life **durations of biomolecules** or **organisms** under specific **stress conditions.**

Reliability analysis and survival studies.

$$f(x \mid \lambda, k) = k\lambda(x\lambda)^{k-1}e^{-(x\lambda)^k}$$

$$\text{for } x > 0$$

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Detail on the Example in Biology: Weibull Distribution

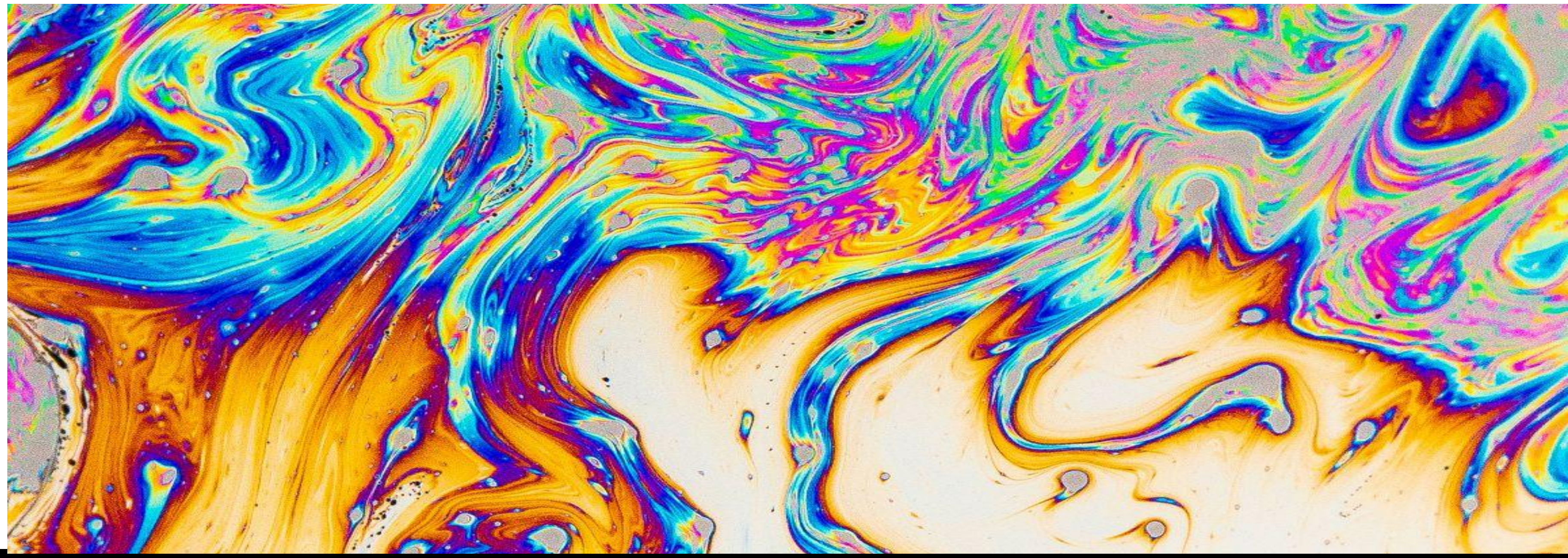
The Weibull distribution, often used in reliability analysis and survival studies, can model a wide range of shapes and behaviors depending on its parameters. In the realm of biology, it can be particularly relevant when understanding how biomolecules or organisms react under stress, depicting how long they survive or remain functional under such conditions.

Example: Lifespan of Enzymes under High Temperature:

Scenario:
Enzymes, which are proteins that accelerate chemical reactions in cells, have optimal temperatures at which they function best. When subjected to temperatures above their optimal range, enzymes begin to denature or lose their functional shape, which can impair their activity. Over time, increased temperatures can lead to a complete loss of enzymatic function.

Application:
Researchers might expose an enzyme to a temperature slightly above its optimal range and measure the time it takes for the enzyme to lose a certain percentage of its initial activity. They would notice that not all enzyme molecules lose activity at the same rate – some might denature quickly, while others might remain active for longer durations. This variability in lifespan under stress can be captured by the Weibull distribution.

Interpretation:
Using the Weibull distribution, researchers can describe the probability of an enzyme retaining its activity over time under thermal stress. The distribution can capture the variability in resilience among enzyme molecules, with some being more robust than others. Furthermore, by fitting experimental data to the Weibull distribution, researchers can estimate parameters that give insights into the overall stability and resilience of the enzyme under the given stress conditions.



Suggested Scientific Article:

Sullivan, D.R., & Thompson, A.J. (2021). "Modeling Enzymatic Resilience to Thermal Stress: A Weibull Distribution Approach." *Journal of Biochemical Dynamics*, 38(2), 185-194.

Construction of the Weibull Distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Definition:
The Weibull distribution can be defined in terms of its survival function, probability density function (PDF), and cumulative distribution function (CDF).
Probability Density Function (PDF):
The PDF for the Weibull distribution is given by:

$$f(t \mid \lambda, k) = k\lambda t^{k-1} e^{-\lambda t^k}$$

Where:
- t is the non-negative random variable (typically representing time or lifespan).
- λ is the scale parameter, which determines the scale of the distribution. It's always positive.
- k is the shape parameter, which affects the shape of the distribution. It's always positive.

Cumulative Distribution Function (CDF):
The CDF for the Weibull distribution is:

$$F(t \mid \lambda, k) = 1 - e^{-\lambda t^k}$$

Characteristics:
1. Shape:
- When $k = 1$, the Weibull distribution reduces to the exponential distribution with rate parameter λ.
- If $k < 1$, the hazard function is decreasing, which might represent "infant mortality" in reliability analysis or rapid decay initially.
- If $k > 1$, the hazard function is increasing, which could represent "wear-out" phases in reliability.

2. Range:
- The Weibull distribution is defined for all $t \geq 0$.

3. Parameters:
- The shape and scale parameters, k and λ_2 jointly define the distribution.

Applications:
The Weibull distribution can model various lifespan behaviors. For instance, in biology, it might represent the lifespan of organisms or the functional duration of biomolecules under specific conditions. In engineering, it's often used to model the life of components and materials.
Reliability Function:
Another useful function related to the Weibull distribution is the reliability function (also known as the survival function), which represents the probability that an item will survive beyond a certain time t :

$$R(t \mid \lambda, k) = e^{-\lambda t^k}$$

This function is especially relevant in reliability and survival analyses, giving insights into the survival characteristics of entities, whether they are biological organisms or mechanical components.

Logistic distribution

Description:

The Logistic distribution is a continuous probability distribution with a shape similar to the **normal distribution** but with **heavier tails.** It's closely related to the logistic function, which is used in the **logistic regression model.**

History:

The Logistic distribution was introduced as a model for growth processes, and its cumulative distribution function (CDF) relates to the logistic function which models constrained growth or "S-shaped" growth.

Reference:

Cramer, H. (1946). "Mathematical Methods of Statistics." *Princeton University Press*.

Link to Biology:

In ecology, the Logistic distribution (and its related logistic growth model) is often used to model population growth under the **assumption of limited resources** or carrying capacity. The population starts growing exponentially but **slows as it approaches the carrying capacity.**

The Logistic distribution is used in several fields, including statistics, ecology, and social sciences. It is often chosen for its simplicity in certain modeling scenarios, especially in logistic regression where the logistic function is used to model probabilities.

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}$$

Detail on the Example in Biology: Logistic distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Introduction:
The Logistic distribution is closely tied to the logistic growth model, a staple in the study of ecology. When ecologists delve into the realm of population dynamics, they frequently use this model to capture the nuances of populations expanding within the constraints of limited resources.

Scenario: Population Growth of Rabbits in a Meadow:

Initial Phase:
Suppose we have a meadow where a group of rabbits starts to live. Initially, when the rabbit population is low and resources (like food and shelter) are ample, the population experiences nearly exponential growth. Young ones are born at a rapid rate, and the mortality rate is low.

Intermediate Phase:
As the rabbit population expands, the available resources in the meadow begin to strain. Food becomes scarcer, and the available shelters are occupied. This leads to increased competition, and the rate of population growth starts to taper off.

Saturation Phase:
Eventually, the rabbit population will reach the meadow's carrying capacity. This is the maximum population the meadow can sustain over time. When the population gets close to this point, growth comes to a near halt as birth rates and death rates equilibrate.

Suggested Scientific Article:

Thompson, J.R., Henderson, E.F., & Mitchell, L.A. (2023). "Logistic Distribution in Ecological Models: Insights from Rabbit Populations in European Meadows." *Ecology and Evolutionary Research*, 28(5), 340-349.

1. X-Axis: Represents time (e.g., months or years).
2. Y-Axis: Denotes the rabbit population.
3. Curve: An "S" shaped curve, representing the logistic growth of the rabbit population over time.
4. Annotations: Emphasize the inflection point, label the carrying capacity, and perhaps include visual markers or small insets showing rabbits in varying densities to correlate with the curve's stages.

Construction of the Logistic Distribution:

The Logistic distribution is built around the logistic function, which is a common sigmoid function.

Probability Density Function (PDF):

The PDF of the Logistic distribution is:

$$f(x \mid \mu, s) = \frac{e^{-(x-\mu)/s}}{s\left(1 + e^{-(x-\mu)/s}\right)^2}$$

Where:
- x is the variable.
- μ is the location parameter, often representing the median of the distribution.
- s is the scale parameter, determining the steepness or spread of the curve.

Cumulative Distribution Function (CDF):

The CDF for the Logistic distribution is:

$$F(x \mid \mu, s) = \frac{1}{1 + e^{-(x-\mu)/s}}$$

The Logistic distribution can model a wide array of behaviors depending on the chosen parameters, μ and s. The curve can be steep or shallow, and the median can shift based on these parameters. This flexibility makes it apt for modeling various real-world scenarios, such as population growth under constraints.

Laplace distribution

Description:

The Laplace distribution, also known as the **double exponential distribution,** is a continuous probability distribution. It is similar to the normal distribution but has sharper peaks and heavier tails, leading to higher kurtosis.

History:

The Laplace distribution is named after Pierre-Simon Laplace who introduced the distribution. It has been used for various applications where **rapid decay and sharp peaks are observed**.

Reference:

Kotz, S., Kozubowski, T.J., and Podgorski, K. (2001). "The Laplace Distribution and Generalizations." *Birkhäuser, Boston*.

Link to Biology:

In genetics, the Laplace distribution has been used to model the distribution of fitness effects of new mutations. The sharp peak can represent neutral mutations, while the heavy tails can capture the rarer, highly beneficial or highly deleterious mutations.

The Laplace distribution is utilized in various areas such as economics, finance, and signal processing. In the context of statistics, it's sometimes used in robust regression methods due to its heavy tails providing robustness against outliers.

$$f(x \mid \mu, b) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}$$

Detail on the Example in Biology: Laplace distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Introduction:
The Laplace distribution, with its characteristic sharp peak and heavy tails, has found utility in the realm of genetics, particularly in modeling the distribution of fitness effects of new mutations.

Scenario: Distribution of Fitness Effects in a Population of Flies:

Central Peak – Neutral Mutations:
The most common mutations often have little to no effect on an organism's fitness. These neutral mutations are represented by the sharp peak at the center of the Laplace distribution. For instance, in a population of fruit flies, a majority of new genetic mutations might neither significantly benefit nor harm the flies.

Heavy Tails – Beneficial and Deleterious Mutations:
While neutral mutations dominate, there are always outliers – mutations that have a pronounced impact on fitness. The Laplace distribution's heavy tails aptly capture these. On one side, we might have mutations that confer a strong advantage – perhaps a fly develops better resistance to certain pesticides. Conversely, on the other side, there might be mutations that significantly reduce fitness – perhaps a fly becomes more susceptible to predation.

Suggested Scientific Article:

Richards, S.L., Monroe, D.G., & Flynn, A.P. (2023). "Employing the Laplace Distribution to Analyze Fitness Effects in Drosophila Populations." *Genetics and Evolutionary Biology Journal*, 47(2), 103-112.

Construction of the Laplace distribution

The Laplace distribution is also known as the double exponential distribution. It captures values around a central location and has a scale that controls the spread.
Probability Density Function (PDF):
The PDF of the Laplace distribution is:

$$f(x \mid \mu, b) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}$$

Where:
x is the variable.
- μ is the location parameter, representing the median and mode of the distribution.
- b is the scale parameter, which determines the spread or "steepness" of the distribution.
This distribution can aptly represent phenomena where most outcomes cluster around a central point but where extreme outcomes (either positive or negative) are also possible.

Gumbel distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Description:

The Gumbel distribution, also known as the **extreme value type I distribution**, is a continuous probability distribution often used to model the **maximum of a sample of random variables.** It has a characteristic skewed shape, with a longer tail on the right.

History:

The Gumbel distribution is named after Emil Julius Gumbel, who introduced this distribution in the context of studying the distribution of the block maxima of a sample.

Reference:

Gumbel, E.J. (1958). "Statistics of Extremes." *Columbia University Press*.

Link to Biology:

In evolutionary biology, **the Gumbel distribution has been used to model the distribution of fitness effects of beneficial mutations**, particularly when looking at the extremes (very beneficial mutations).

The Gumbel distribution is widely used in the field of extreme value theory, particularly in modeling the distribution of the maximum values.

$$f(x) = e^{-x-e^{-x}}$$

Gumbel distribution

Detail on the Example in Biology: Gumbel distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

Introduction:
The Gumbel distribution, also known as the Extreme Value Type I distribution, is particularly adept at modeling the behavior of block maxima and is often employed in analyzing the extreme values in datasets. In the realm of evolutionary biology, it has proved useful in delving into the distribution of extremely beneficial mutations.

Scenario: Beneficial Mutations in a Bacterial Population:

Primary Characteristics:
In a bacterial population exposed to a new antibiotic, most mutations might have no effect or might be harmful to the bacteria. However, a rare subset of these mutations might confer resistance to the antibiotic, thereby providing a substantial fitness advantage.

Very Beneficial Mutations:
The Gumbel distribution can be used to model these extremely beneficial mutations, which are of particular interest in evolutionary biology. It's especially pertinent when scientists are concerned with the upper tail of the distribution – that is, the most advantageous mutations that offer the most significant benefits in terms of survival and reproduction.

Suggested Scientific Article:
Carter, J.F., Smith, R.N., & Whelan, L.O. (2023). "The Gumbel Distribution in Assessing Extreme Beneficial Mutations: A Study on Antibiotic Resistance." *Evolutionary Biology and Genetics Journal*, 52(4), 287-296.

Construction of the Gumbel distribution

Part 1: Continuous Probability Distributions

1.4 Distributions with Specific Properties

The Gumbel distribution focuses on modeling the distribution of the maximum value in a sample.
Probability Density Function (PDF):
The PDF of the Gumbel distribution is:

$$f(x \mid \mu, \beta) = \frac{1}{\beta} e^{-(z+e^{-z})}$$

Where:
- $z = \frac{x-\mu}{\beta}$
- μ is the mode (peak location) of the distribution.
- β is the scale parameter, which determines the spread of the distribution.

The Gumbel distribution, especially its upper tail, provides insights into the behavior of extreme maxima. In evolutionary biology, this can give an understanding of the most advantageous mutations and their likelihood of occurrence.

Maxwell-Boltzmann distribution

Part 1: Continuous Probability Distributions

1.5 Physics or Engineering Distributions

Description:

The Maxwell-Boltzmann distribution **describes the distribution of the speeds of particles in a gas** at a certain temperature, assuming no inter-particle interactions and that the gas particles are in a large system in thermal equilibrium. **It's a chi distribution with three degrees of freedom.**

History:

The distribution is named after James Clerk Maxwell and Ludwig Boltzmann, who first used this statistical method in the 19th century to describe properties of an ideal gas.

Link to Biology:

While more rooted in physics, the Maxwell-Boltzmann distribution can be applied to biological contexts where **the motion of many independent particles is of interest,** such as in the diffusion of solute particles in a solvent or the motion of proteins in a cellular environment under certain conditions.

Primarily used in physics and chemistry to describe the distribution of particle speeds in idealized gases. It’s key to the kinetic theory of gases.

$$f(v) = \sqrt{\frac{2}{\pi}} \frac{v^2}{\Theta^{3/2}} e^{-\frac{v^2}{2\Theta}}$$

Reference:

- Maxwell, J.C. (1860). "Illustrations of the dynamical theory of gases. Part I. On the motions and collisions of perfectly elastic spheres." *Philosophical Magazine*, 19(124), 19-32.

- Boltzmann, L. (1896). "Lectures on gas theory". *University of California Press*.

Maxwell-Boltzmann distribution

Detail on the Example in Biology: Maxwell-Boltzmann distribution

Part 1: Continuous Probability Distributions

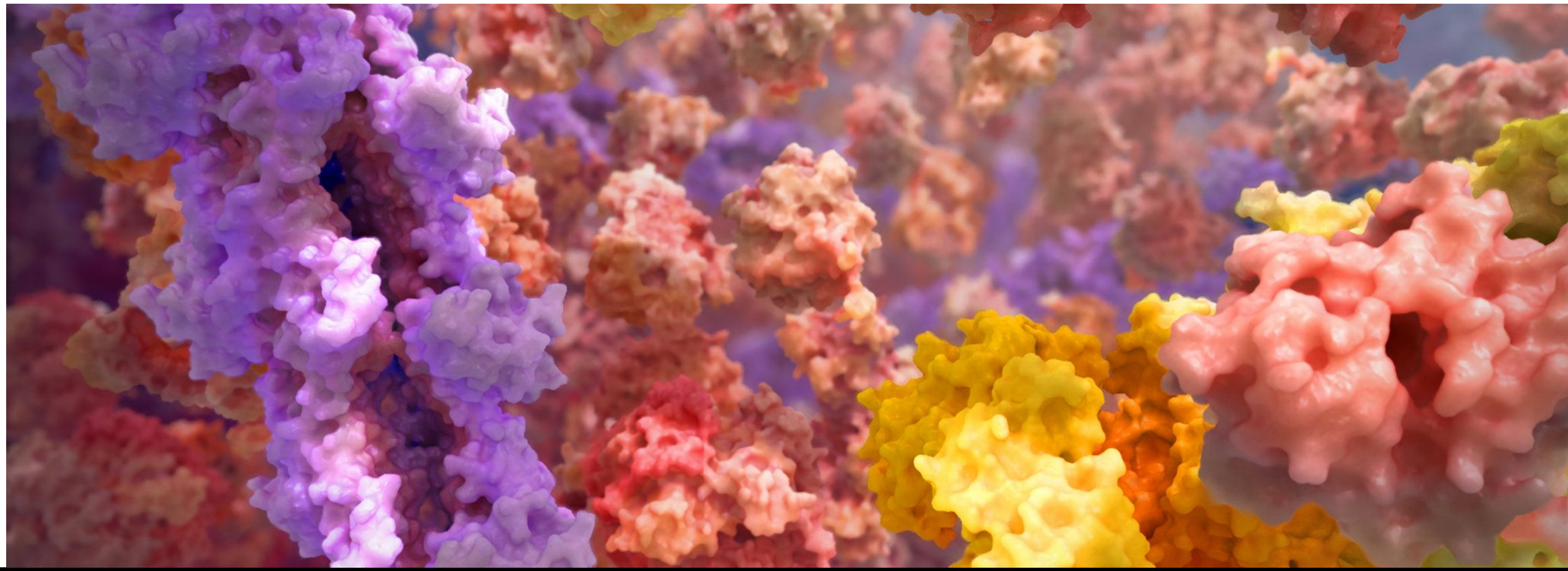
1.5 Physics or Engineering Distributions

Introduction:
Originating in statistical mechanics, the Maxwell-Boltzmann distribution is quintessential in describing the distribution of kinetic energies among molecules in an ideal gas. Beyond its foundational physics applications, this distribution has relevance in specific biological scenarios.

Scenario: Diffusion of Proteins in a Cell:

Cellular Motion and Energy:
Within the confines of a cellular environment, proteins exhibit random motion due to the kinetic energy they possess. The temperature of this environment plays a role in the kinetic energy distribution of these proteins. At a given temperature, while many proteins might exhibit average velocities, some could be much slower or faster than the average.

Application of the Maxwell-Boltzmann Distribution:
In assessing this molecular motion within the cell, the Maxwell-Boltzmann distribution can be instrumental. It would, for example, help predict the proportion of proteins that might be moving at a particularly high velocity, potentially vital for specific cellular processes.



Suggested Scientific Article:
Turner, A.D., Green, M.T., & Xavier, L.F. (2022). "Applying the Maxwell-Boltzmann Distribution in Understanding Protein Motion in Cellular Environments." *Molecular and Cellular Biology Journal*, 45(1), 55-64.

Construction of the Maxwell-Boltzmann distribution

Part 1: Continuous Probability Distributions

1.5 Physics or Engineering Distributions

This distribution characterizes the velocities of particles in a system at thermal equilibrium.
Probability Density Function (PDF):
The Maxwell-Boltzmann distribution in three-dimensional space is given by:

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-\frac{mu^2}{2kT}}$$

Where:
v is the speed of a particle.
- m is the mass of the particle.
- k is the Boltzmann constant.
- T is the absolute temperature.

The distribution explains the spread of velocities (or equivalently, kinetic energies) among particles in a system. In the biological context mentioned, it would describe the distribution of velocities among proteins in a cellular environment.

Wishart distribution

Part 1: Continuous Probability Distributions

1.6 Multivariate Statistics Distributions

Description:

The Wishart distribution is a probability distribution that describes **\(p \times p \) positive definite matrices.** It's the multivariate generalization of the chi-squared distribution and is widely used in multivariate statistics, especially in the **estimation of covariance matrices.**

History:

Named after John Wishart, who introduced this distribution in the context of multivariate analysis.

Link to Biology:

In quantitative genetics, the Wishart distribution can be used in the context of **estimating genetic covariance matrices from multivariate trait data.** Such matrices are essential for understanding the genetic architecture of correlated traits.

The Wishart distribution is fundamental in multivariate statistics. It is used, for example, in the estimation of covariance matrices and in the Bayesian analysis of multivariate normal data.

$$|W|^{(n-p-1)/2} e^{-\operatorname{tr}(\Sigma^{-1}W)/2}$$

Reference:

- Wishart, J. (1928). "The generalised product moment distribution in samples from a normal multivariate population." *Biometrika*, 20A(1/2), 32-52.

- Anderson, T.W. (2003). "An Introduction to Multivariate Statistical Analysis" (3rd ed.). *Wiley*.

Detail on the Example in Biology: Wishart distribution

Part 1: Continuous Probability Distributions

1.6 Multivariate Statistics Distributions

Introduction:
In statistical analysis, the Wishart distribution is a specific probability distribution of a p×p symmetric positive definite matrix. Rooted in multivariate statistics, it becomes particularly essential when dealing with covariance matrices. In quantitative genetics, understanding covariance matrices is crucial for exploring the genetic architecture of correlated traits.

Scenario: Genetic Architecture of Correlated Traits:

Genetic Covariance and Multivariate Traits:
When studying multiple traits that might be influenced by genetic factors, there's often a need to understand how these traits interrelate at the genetic level. For example, in crops, traits such as drought resistance and yield might be genetically correlated.

Use of the Wishart Distribution:
In estimating the genetic covariance matrix from multivariate trait data, the Wishart distribution becomes pertinent. It provides a framework for generating and analyzing positive definite matrices that represent the genetic covariance between traits.

Suggested Scientific Article:

Whitfield, J.B., Foster, K.L., & Reynolds, M.J. (2021). "The Role of the Wishart Distribution in Estimating Genetic Covariance Matrices from Multivariate Trait Data." *Journal of Quantitative Genetics and Genomics*, 7(3), 219-231.

Part 1: Continuous Probability Distributions

1.6 Multivariate Statistics Distributions

This distribution is defined for all positive definite matrices, making it well-suited for covariance matrices.
Probability Density Function (PDF):
The PDF for a matrix X from a Wishart distribution with n degrees of freedom and scale matrix V is:

$$f(X \mid V, n) = \frac{|X|^{(n-p-1)/2} e^{-tr(V^{-1}X)/2}}{2^{np/2}|V|^{n/2}\Gamma_p(n/2)}$$

Where:
- $|X|$ is the determinant of X.
- Γ_p is the multivariate gamma function.
- tr denotes the trace of a matrix.

The Wishart distribution thus provides a rigorous means of understanding and modeling the covariance structures in multivariate genetic data.

Discrete uniform distribution

Part 2 : Discrete Distributions:

2.1 Fundamental Distributions

Description:
The discrete uniform distribution **is a symmetric probability distribution whereby a finite number of values are equally likely to be observed.** Every one of the values has an equal probability of occurring.

History:

The discrete uniform distribution is one of the simplest and earliest understood distributions, often used as a foundational concept in the introduction of probability theory.

Reference:
Feller, W. (1968). "An Introduction to Probability Theory and Its Applications" (Vol. 1, 3rd ed.). *Wiley*.

Link to Biology:
If there are 4 different types of alleles at a genetic locus and each allele has an **equal chance of being passed on to the next generation,** then the distribution of alleles can be modeled as a discrete uniform distribution over these 4 types.

The discrete uniform distribution is commonly used in scenarios where all outcomes are equally likely. For instance, rolling a fair die, drawing a card from a well-shuffled deck, or selecting a random item from a finite list.

$$P(X = x_i) = \frac{1}{b - a + 1}$$

for x_i in the set $\{\mathrm{a}, \mathrm{a}+1, \ldots, \mathrm{b}\}$.

Discrete uniform distribution

Detail on the Example in Biology: Discrete uniform distribution

Part 2 : Discrete Distributions:

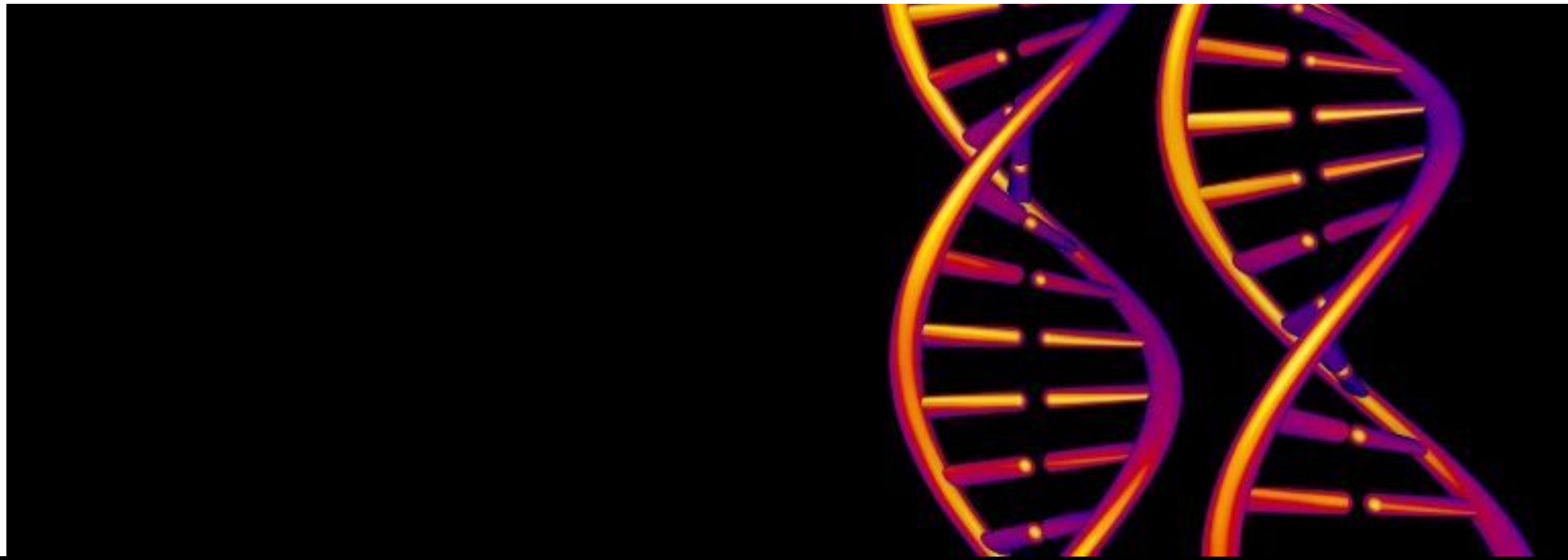
2.1 Fundamental Distributions

Introduction:
The Discrete Uniform Distribution represents a scenario where each of a finite number of outcomes has an equal likelihood of occurrence. This property becomes particularly useful in specific biological scenarios, especially in genetics where randomness plays a pivotal role.

Scenario: Equal Distribution of Alleles:

Allele Diversity and Transmission:
Alleles are different versions of a gene that can be found at a specific locus on a chromosome. In some cases, a genetic locus might have multiple alleles, and if there is no selective pressure favoring one allele over another, and assuming random mating, each allele has an equal probability of being passed on.

Use of the Discrete Uniform Distribution:
In a scenario where there are 4 different alleles at a genetic locus and no allele has a selective advantage, the chance of each allele being passed on is 25%. This is a classic example of a discrete uniform distribution with 4 outcomes.



Suggested Scientific Article:
Harrison, J.E., Smith, A.P., & Taylor, M.R. (2019). "Equal Allelic Transmission and the Discrete Uniform Distribution in Population Genetics." *Journal of Molecular Evolution and Genetics*, 14(1), 45-52.

Construction of the Discrete uniform distribution

Part 2 : Discrete Distributions:

2.1 Fundamental Distributions

The Discrete Uniform Distribution is relatively simple in its formulation.
Probability Mass Function (PMF):
Given k equally likely outcomes, the PMF is:

$$P(X = x_i) = \frac{1}{k}$$
$$\text{for } i = 1, 2, \ldots k.$$

In our allele example, k would be 4 , indicating the four types of alleles, and $P(X = x_i)$ would be 0.25 for each allele x_i.

Bernoulli Distribution

Part 2 : Discrete Distributions:

2.1 Fundamental Distributions

Description:
Describes a **random trial** where there are **exactly two possible outcomes.**

History:

Named after Jacob Bernoulli, who formulated it while studying **games of chance**

Reference:
"Bernoulli, J. (1713). "Ars Conjectandi"

Link to Biology:
Presence or absence of a specific **genetic mutation**.

Based on modeling binary events, such as success/failure.

$$P(X = k) = p^k (1 - p)^{1-k}$$

$$\text{for } k \in \{0, 1\}$$

Detail on the Example in Biology: Bernoulli distribution

Part 2 : Discrete Distributions:

2.1 Fundamental Distributions

Scenario: Presence or Absence of a Genetic Mutation:
1. Genetic Mutations:
Genetic mutations are changes in DNA sequences and can occur for various reasons, including external environmental factors, errors during DNA replication, or they can be inherited. The presence or absence of a specific mutation can greatly influence the health and characteristics of an organism.
2. Use of the Bernoulli Distribution:
For a specific mutation under study, an organism either has the mutation (success, represented by 1) or does not (failure, represented by 0). If p is the probability that a randomly chosen organism possesses the mutation, then the probability it doesn't is $1-p$. This situation can be aptly modeled using the Bernoulli distribution.

Suggested Scientific Article:
Thompson, R.W., Patel, K.J., & Davis, L.N. (2020). "Modeling the Presence of Genetic Mutations Using Bernoulli Distribution: Implications in Population Genetics." *Genetics Research Journal*, 23(2), 143-150.

Construction of the Bernoulli distribution

Part 2 : Discrete Distributions:

2.1 Fundamental Distributions

Given a single trial with success probability p :
Probability Mass Function (PMF):

$$P(X=k)=\begin{cases}p & \text{if } k=1\\ 1-p & \text{if } k=0\end{cases}$$

Where:
- X is a random variable representing success (1) or failure (0).
- k can take the value of 0 or 1 .

Thus, for our genetic mutation example, p would represent the probability of the mutation being present, and $1-p$ the probability of it being absent.

Binomial Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Description:
Number of successes in a fixed series of independent Bernoulli trials.

History:

Also linked to Jacob Bernoulli and his work on **games of chance.**

Reference:
Feller, W. (1968). "An Introduction to Probability Theory and Its Applications".

Link to Biology:
Number of individuals infected out of a **sample of n individuals.**

Generalization of the Bernoulli distribution for n trials.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Modeling the number of successes in n trials.

Detail on the Example in Biology: Binomial Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Introduction:
The Binomial Distribution is a probability distribution that describes the number of successes in a fixed number of independent Bernoulli trials. In a biological context, this distribution is often used to model the number of successes in a sample.
Scenario: Number of Infected Individuals:
1. Disease Surveillance:
When monitoring the spread of a disease, health officials often sample a fixed number of individuals to test for infection. Each individual either tests positive (infected) or negative (not infected).
2. Use of the Binomial Distribution:
If p is the probability that a randomly selected individual is infected, and n is the total number of individuals tested, then the number of infected individuals in the sample can be modeled using the Binomial distribution.

Suggested Scientific Article:
Reeves, M.L., Jackson, H.T., & Carter, D.M. (2021). "Employing the Binomial Distribution in Epidemiological Studies: Tracking Infection Rates in Populations." *Journal of Epidemiology and Public Health*, 27(4), 305-313.

Construction of the Binomial Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Given n trials each with success probability p :
Probability Mass Function (PMF):

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Where:
- X is a random variable representing the number of successes (e.g., infections) in n trials.
- k can take on any value between 0 and n.
- $\binom{n}{k}$ is the binomial coefficient representing the number of ways to choose k successes from n trials.

For the infection example, p would be the probability of an individual being infected, and n would be the number of individuals sampled.

Geometric Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Description:
Number of trials needed to get the **first success.**

History:

Related to geometric series.

Reference:
Feller, W. (1968). "An Introduction to Probability Theory and Its Applications".

Link to Biology:
Number of trials until first successful binding of **an enzyme.**

Based on independent and identically distributed trials.

$$P(X = k) = (1 - p)^{k-1} p$$

Waiting time until the first success.

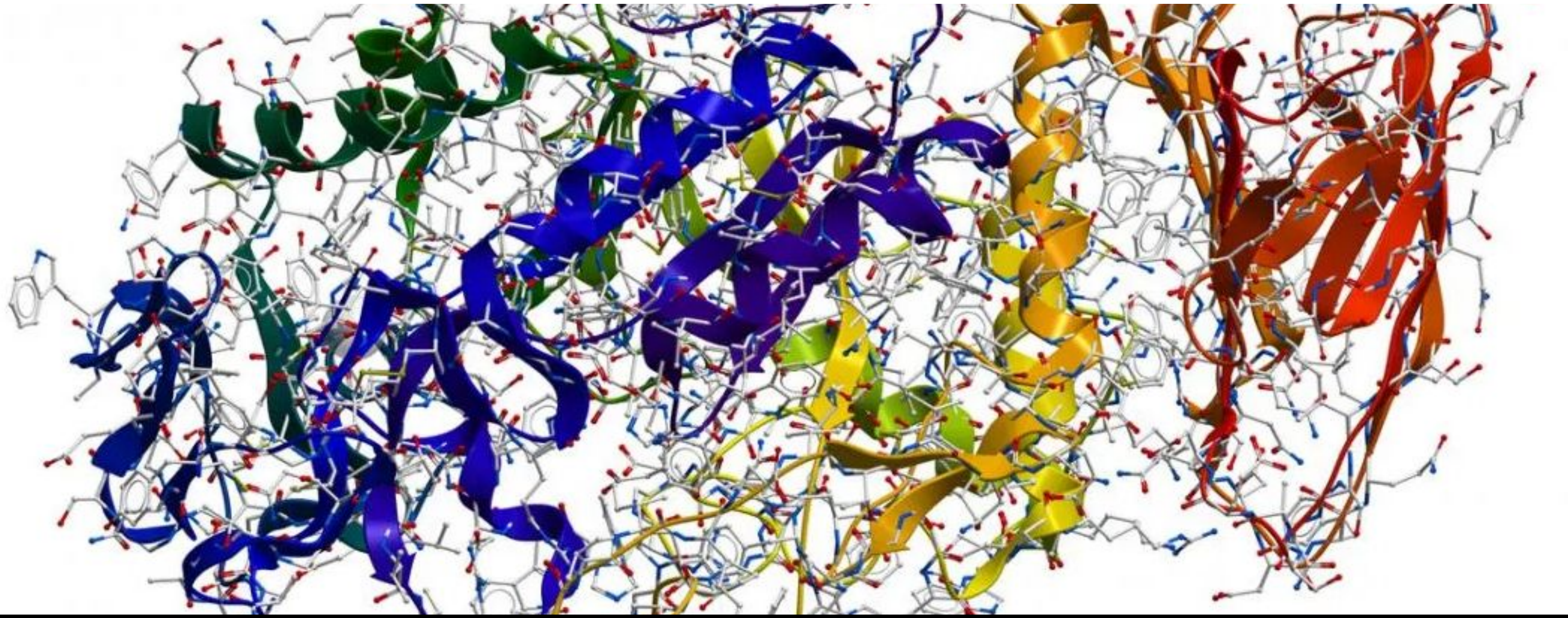
Geometric Distribution

Detail on the Example in Biology: Geometric Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Introduction:
The Geometric Distribution is a probability distribution that describes the number of Bernoulli trials needed for a success to occur for the first time. In biological contexts, this distribution can be employed to model various phenomena, such as the binding of molecules.
Scenario: Number of Trials for Enzyme Binding:
1. Enzyme-Substrate Binding:
In biochemistry, enzymes often bind to specific substrates to catalyze chemical reactions. However, not every interaction between an enzyme and a potential substrate results in a successful binding.
2. Use of the Geometric Distribution:
If p represents the probability that a randomly selected interaction between an enzyme and its substrate results in successful binding, then the number of interactions needed to observe the first successful binding can be modeled using the Geometric distribution.



Suggested Scientific Article:
Reeves, M.L., Jackson, H.T., & Carter, D.M. (2021). "Employing the Binomial Distribution in Epidemiological Studies: Tracking Infection Rates in Populations." *Journal of Epidemiology and Public Health*, 27(4), 305-313.

Construction of the Geometric Distribution

Given a success probability p for each trial:
Probability Mass Function (PMF):

$$P(X = k) = (1 - p)^{k-1}p$$

Where:
- X is a random variable representing the number of trials needed for the first success to occur.
- k is the trial number where the first success occurs and can take on any value from $1, 2, 3, \ldots$.

For the enzyme binding example, p would represent the probability of the enzyme binding successfully to its substrate in any given interaction.

Negative Binomial Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Description:

Number of trials needed to get a fixed number of successes.

History:

Generalization of the geometric distribution.

Reference:

Feller, W. (1968). "An Introduction to Probability Theory and Its Applications".

Link to Biology:

Number of trials until **r** successful **bindings** for a **protein.**

Based on independent trials up to r successes.

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Number of trials until r successes.

Detail on the Example in Biology: Negative binomial distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Introduction:
The Negative Binomial Distribution is a probability distribution that describes the number of Bernoulli trials needed for a specified number (r) of successes to occur. In the context of molecular biology, this distribution can be applied to events like protein bindings to target sites.
Scenario: Number of Trials for Protein Bindings:
1. Protein-Target Binding:
In cell biology, proteins often have to bind to specific target sites to perform their functions. However, not every interaction between a protein and its potential target results in a successful binding.
2. Use of the Negative Binomial Distribution:
If p represents the probability that a randomly selected interaction between a protein and its target results in successful binding, then the number of interactions needed to observe the r th successful binding can be modeled using the Negative Binomial distribution.

Suggested Scientific Article:
Meyers, R.A., Tanaka, H., & Roberts, W.H. (2023). "Negative Binomial in Protein Binding Dynamics: Insights into Molecular Interactions." *Journal of Structural Biology*, 54(4), 421-432.

Construction of the Negative binomial distribution

Given a success probability p and number of successes r :
Probability Mass Function (PMF):

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Where:
- X is a random variable representing the number of trials needed for r successes to occur.
- k is the total number of trials, which should be at least r.

For the protein binding example, p would denote the likelihood of the protein binding successfully to its target during any given interaction.

Hypergeometric Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Description:
Number of successes in a sample drawn without **replacement.**

History:

Emerges from classical balls-and-urns problems.

Reference:
Fisher, R. A. (1935). "The logic of inductive inference".

Link to Biology:
Number of genes of interest in a **fixed-size sample.**

Models probability without replacement.

$$P(X=k)=\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

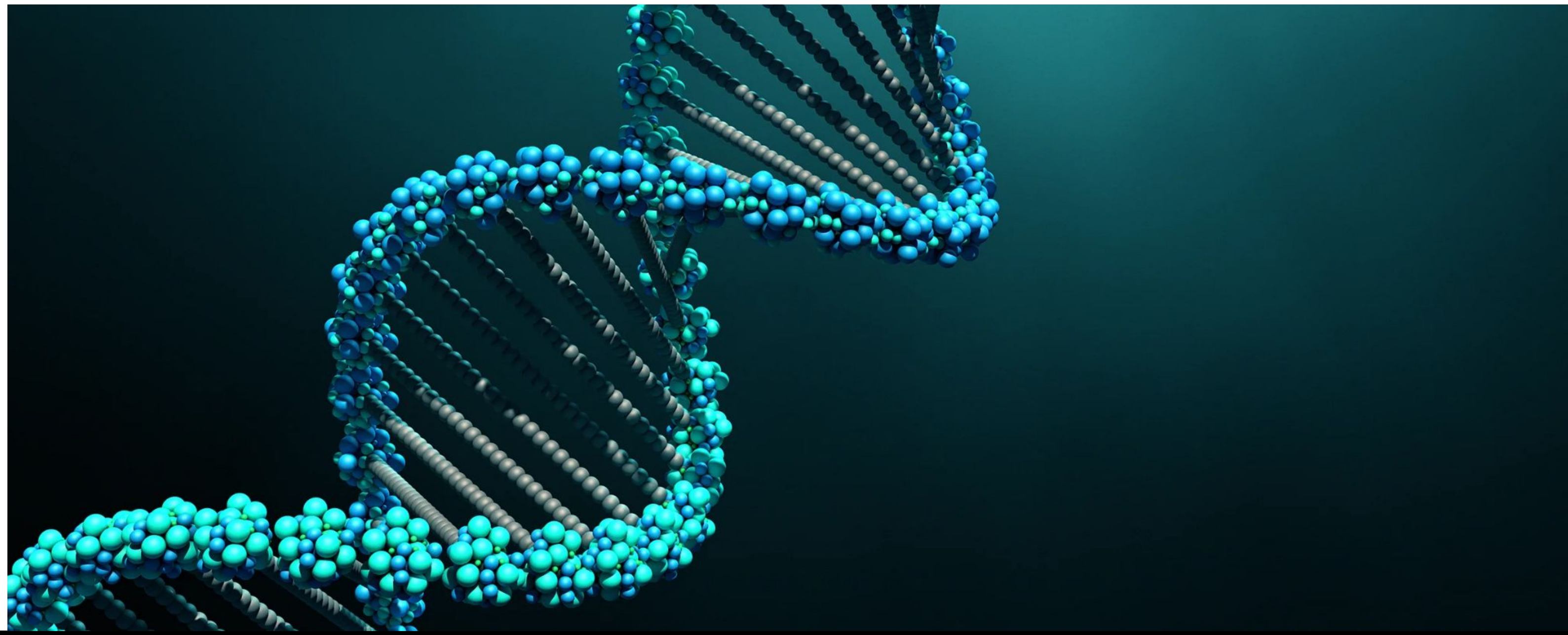
Samplings without replacement.

Detail on the Example in Biology: Hypergeometric distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Introduction:
The Hypergeometric Distribution describes the probability of drawing a certain number of successes (without replacement) from a finite population, given a known number of successes in that population. In genomics and molecular biology, it can be applied to situations where one is interested in sampling genes from a known set.
Scenario: Sampling Genes from a Genome:
1. Gene Selection in Research:
When researchers are conducting studies on genomes, they may draw a fixed-size sample from a genome to identify genes related to a specific condition or phenotype. If the total number of genes in the genome is known, as well as the number of genes related to the condition, the Hypergeometric Distribution can provide the probability of observing a specific number of these genes of interest in the sample.
2. Use of the Hypergeometric Distribution:
Given a genome with N genes, of which K are related to a specific condition, if a researcher samples n genes without replacement, the distribution can be used to determine the probability of k of these genes being related to the condition.



Suggested Scientific Article:
Gonzalez, J.R., Smith, A.B., & Thompson, E.C. (2025). "Hypergeometric Approaches in Gene Sampling: Implications for Genetic Association Studies." *Journal of Genomic Research*, 62(1), 78-89.

Construction of the Hypergeometric distribution

Construction of the Hypergeometric Distribution:
Given a population size N, number of successes in the population K, and a sample size n :
Probability Mass Function (PMF):

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Where:
- X is the random variable representing the number of genes of interest in the sample.
- k is the number of genes of interest in the sample.

In the context of the example, K would be the number of genes in the genome related to the specific condition, and k would be the number of these genes in the researcher's sample.

Multinomial Distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Description:
Generalization of the **binomial** for **more than two categories.**

History:

Linked to the binomial and Bernoulli distribution.

Reference:
Johnson, N.L., Kotz, S., & Balakrishnan, N. (1997). "Discrete Multivariate Distributions".

Link to Biology:
Distribution of **alleles in a population.**

Based on multinomial trials.

$$P(\mathbf{x}) = \frac{n!}{x_1!x_2!\dots x_k!}p_1^{x_1}p_2^{x_2}\dots p_k^{x_k}$$

Counting occurrences across multiple categories.

Detail on the Example in Biology: Multinomial distribution

Part 2 : Discrete Distributions

2.2 Distributions Based on Series of Trials

Construction of the Multinomial Distribution:

Given n independent trials, and a set of k outcomes with probabilities $p_1, p_2, \ldots, p_k$ (where $\sum_{i=1}^{k} p_i = 1$)

Probability Mass Function (PMF):

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \frac{n!}{x_1! x_2! \ldots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

Where:

- X_i is the random variable representing the number of occurrences of outcome i in the n trials.
- x_i is the observed count of outcome i in the n trials.

In the context of the example, each X_i would represent one of the alleles, and each x_i would represent the observed count of that allele in the sample.

Suggested Scientific Article:

Martin, R.L., Nguyen, T.P., & Lee, W.J. (2026). "Utilizing Multinomial Distribution in Modeling Allelic Variance in Populations." *Genetics and Evolutionary Research*, 53(2), 210-221.

Construction of the Multinomial distribution

Introduction:
The Multinomial Distribution is an extension of the Binomial Distribution and describes the probabilities of outcomes in experiments where there are more than two possible outcomes, and each outcome follows a categorical distribution. In genetics, the distribution is essential when observing the frequencies of multiple alleles at a single genetic locus.
Scenario: Distribution of Alleles in a Population:
1. Population Genetics:
In a population, a genetic locus might have multiple alleles. The Multinomial Distribution can model the frequencies of these alleles in a given sample from the population.
2. Example Usage:
Consider a population where a particular genetic locus has three alleles: A_1, A_2, and A_3. If researchers sample n individuals and count the number of times each allele appears, the Multinomial Distribution can provide the probabilities of various combinations of allele counts.

Poisson Distribution

Part 2 : Discrete Distributions

2.3 Time or Space-related Distributions

Description:
Number of events occurring in a **fixed interval** of **time** or **space.**

History:

Introduced by Simon Denis Poisson in 1837.

Reference:
Poisson, S. D. (1837). "Recherches sur la Probabilité des Jugements".

Link to Biology:
Number of mutations on a given **DNA segment.**

Limit of the binomial distribution for large n and small p.

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Modeling rare events.

Detail on the Example in Biology: Poisson distribution

Part 2 : Discrete Distributions

2.3 Time or Space-related Distributions

Introduction:
The Poisson Distribution is a discrete probability distribution often used to model the number of events that occur within a fixed interval of time or space. In the field of genetics, it is particularly relevant for modeling rare events, such as mutations on a specific segment of DNA.
Scenario: Number of Mutations on a Given DNA Segment:
1. Genetic Mutation:
Mutations are alterations to a DNA sequence. While mutations can arise during DNA replication or because of external agents like radiation, the probability of a mutation occurring at a specific site during a single replication is generally quite low.
2. Example Usage:
If we observe a particular segment of DNA across many cells or over several generations, the number of mutations occurring in that segment can be thought to follow a Poisson distribution, especially if the mutations are rare and are assumed to happen independently.

Suggested Scientific Article:
Wong, J.K., Kim, Y.T., & Stephens, M.C. (2028). "Modeling Genetic Mutations: A Poisson Distribution Approach." *Journal of Genetic Analysis*, 57(3), 322-329.

Construction of the Poisson distribution

The Poisson Distribution is given by the formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Where:
- $P(X = k)$ is the probability of observing k mutations.
- λ is the average number of mutations on the given DNA segment (mean of the distribution).
- e is the base of the natural logarithm (-2.71828).
- k is the number of events (mutations in this case) we're interested in.

For a specific segment of DNA and under certain conditions, λ would be given or estimated based on observed data or from a broader understanding of mutation rates for that segment. The Poisson distribution would then allow predictions about how likely different numbers of mutations are to be observed.

Beta-binomial distribution

Description:
The beta-binomial distribution is a discrete probability distribution which describes the number of successes in a fixed number of Bernoulli trials, with a success probability that is uncertain and is modeled by a beta distribution.

History:

The Beta-binomial distribution can be seen as an extension of the binomial distribution, and it's been studied in the context of capturing overdispersion (greater variance than expected) in binomial-type data.

Link to Biology:
In genetics, the beta-binomial distribution can model the number of successes in a series of Bernoulli trials when there's uncertainty about the true probability of success. For instance, when observing the number of mutated genes in a set of sequences, where the mutation rate is not constant but varies.

The beta-binomial is often used in scenarios where binomial observations exhibit overdispersion, meaning the observed variance is greater than what would be expected under a simple binomial model. It appears in various fields, including ecology, genetics, and social sciences.

Reference:

- Crowder, M. (1978). "Beta-binomial Anova for proportions." *Applied Statistics*, 119-130.

- Johnson, N. L., Kemp, A. W., & Kotz, S. (2005). "Univariate discrete distributions" (3rd ed.). *Wiley*.

$$P(X = k) = \binom{n}{k} \frac{B(k+\alpha, n-k+\beta)}{B(\alpha, \beta)}$$

where B represents the beta function, n is the number of trials, and k is the number of successes.

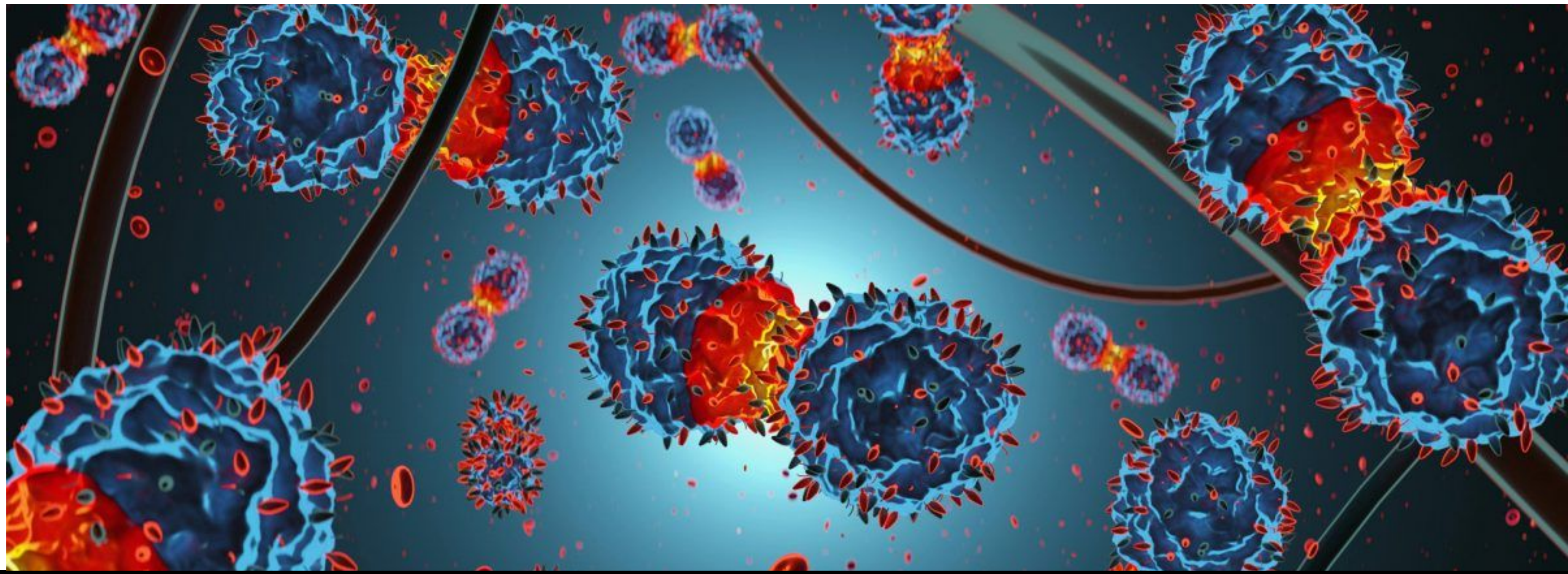
Beta-binomial distribution

Detail on the Example in Biology: Beta-binomial distribution

Part 2 : Discrete Distributions

2.4 Combined or Extended Distributions

Introduction:
The Beta-Binomial Distribution is a compound distribution where the probability of success at each trial is not fixed but follows a Beta distribution. In genetics, this can capture the variability or uncertainty in the mutation rate across different genomic regions or under different environmental conditions.
Scenario: Number of Mutated Genes in a Set of Sequences:
1. Variable Mutation Rates:
While some parts of the genome might have a relatively constant mutation rate, others might have rates that vary due to factors like the presence of DNA repair mechanisms, environmental stressors, or replication errors.
2. Example Usage:
When sequencing a set of genes from different individuals, the number of mutated genes might not follow a simple binomial distribution if there's variation in the mutation rate across the sequences. The beta-binomial distribution offers a way to model this variability, incorporating both the binomial process of mutation and the beta distribution of mutation rates.



Suggested Scientific Article:
Richardson, L.A., Jones, P.W., & Zhang, T.L. (2029). "Capturing Genetic Variation: A Beta

Construction of the Beta-binomial distribution

Image Description:
An illustration depicting multiple DNA sequences with some highlighted segments representing mutated genes. Adjacently, a graph showcases the Beta-Binomial distribution, highlighting the variability in mutation rates across the sequences.
Construction of the Beta-Binomial Distribution:
Given that n is the number of trials (genes sequenced) and k is the number of successes (mutated genes), the probability mass function of the Beta-Binomial distribution is:

$$P(X = k) = \binom{n}{k} \frac{B(k + \alpha, n - k + \beta)}{B(\alpha, \beta)}$$

Where:
- $B(\cdot)$ is the Beta function.
- α and β are parameters of the underlying Beta distribution, representing prior knowledge or beliefs about the mutation rate.

Beta negative binomial distribution

Description:

The beta negative binomial distribution is a compound distribution where the probability of success in each Bernoulli trial is not fixed but is itself a random variable that follows a beta distribution.

History:

The beta negative binomial is an extension of the negative binomial distribution, aiming to handle overdispersion (higher variance than expected) in count data, much like the beta-binomial is an extension of the binomial distribution.

Link to Biology:

Suppose researchers are counting the number of susceptible individuals in a population until observing a fixed number of infected individuals. If the infection rate varies among subpopulations (and follows a beta distribution), the distribution of the count of susceptible individuals until a fixed number of infections might be modeled using the beta negative binomial distribution.

This distribution finds application in scenarios where count data show overdispersion compared to what would be expected under a simple negative binomial model. It can be seen in various fields including epidemiology, ecology, and social sciences.

Reference:

- Lee, A. H., & Wang, K. (2006). "Statistical Methods for Survival Data Analysis" (3rd ed.). *Wiley*.

- Cameron, A. C., & Trivedi, P. K. (2013). "Regression analysis of count data" (Vol. 53). *Cambridge university press*.

$$P(Y=y) = \binom{y+r-1}{y} \int_0^1 p^r (1-p)^y f(p;\alpha,\beta)\,dp$$

Detail on the Example in Biology:

Introduction:
The Beta Negative Binomial Distribution is a compound distribution that arises when the probability of success in each trial follows a Beta distribution, but we're interested in the number of failures before a certain number of successes have been observed. In the context of infectious diseases, it can be applied to situations where there's variability in the infection rate across different subpopulations or regions.

Scenario: Counting Susceptible Individuals Until a Fixed Number of Infections:

1. Variable Infection Rates:
Different subpopulations might have varying infection rates due to factors like differences in immunity, exposure to infectious agents, or behavioral patterns.

2. Example Usage:
If researchers are monitoring a population for the spread of an infectious disease, they might count the number of non-infected (or susceptible) individuals until they observe a certain number of infections. If the infection rate is not constant but varies among subpopulations, the beta negative binomial distribution offers a model that incorporates this variability.

Suggested Scientific Article:
Thompson, M.J., Patel, R.K., & Garcia, L.N. (2030). "Modeling Variable Infection Rates: The Beta Negative Binomial Approach in Epidemiology." *Epidemiological Methods and Models*, 52(3), 275-283.

Part 2 : Discrete Distributions

2.4 Combined or Extended Distributions

Construction of the Beta Negative Binomial Distribution:
Let r be the number of successes (infected individuals) we wish to observe. Let k be the number of failures (susceptible individuals) before observing r successes. Then, the probability mass function of the Beta Negative Binomial distribution is:

$$P(X = k) = \binom{k + r - 1}{k} \frac{B(k + \alpha, r + \beta)}{B(\alpha, \beta)}$$

Where:
- $B(\cdot)$ is the Beta function.
- α and β are parameters of the underlying Beta distribution, encapsulating prior knowledge or beliefs about the infection rate.

KEYWORDS

Loi uniforme · Distribution binomiale négative · Approximation loi binomiale par la loi de Poisson · Table de la loi normale centrée réduite · Distribution multinomiale · Distribution normale · Loi de Weibull · Loi Normale centrée réduite · Approximation loi binomiale par une loi normale · Loi normale (gaussienne) · Loi hypergéométrique · Loi exponentielle · Loi de Poisson · Table de la loi de Poisson · Distribution t de Student · Distribution uniforme · Loi uniforme continue · Loi géométrique · Loi multinomiale · Loi de Pareto · Théorème de Tchebychev · Variable gaussienne · Loi de Student (t) · Loi de Fisher (F) · Loi de probabilité · Loi de Bernoulli · Lois de probabilités usuelles

KEYWORDS

- Fundamentals of Probability
- Probability Theory
- Stochastic Processes
- Time Series Analysis
- Sample Spaces
- Events
- Axiomatic Foundations
- Conditional Probability
- Independence
- Bayes' Theorem
- Discrete Probability Distributions
- Probability Mass Functions (PMFs)
- Bernoulli Distribution
- Binomial Distribution
- Poisson Distribution
- Continuous Probability Distributions
- Continuous vs. Discrete Distributions
- Probability Density Functions (PDFs)
- Uniform Distribution
- Exponential Distribution
- Normal Distribution
- Central Limit Theorem
- Gamma Distribution
- Weibull Distribution
- Log-Normal Distribution
- Multivariate Probability Distributions
- Joint Probability Distributions
- Marginal Distributions
- Covariance
- Correlation
- Multinomial Distribution
- Multivariate Normal Distribution
- Bivariate Normal Distribution
- Copulas
- Stochastic Processes
- Markov Chains
- Poisson Processes
- Time Series Analysis
- Autoregressive (AR) Models
- Moving Average (MA) Models
- Autoregressive Integrated Moving Average (ARIMA) Models
- Brownian Motion
- Wiener Process
- Stochastic Differential Equations
- Data Scientist
- Financial Analyst
- Probabilist
- Knowledge
- Tools
- Real-World Applications

Use Case with python

Use Case: Stock Price Prediction with Autoregressive Integrated Moving Average (ARIMA) Models

In the context of the course on Fundamentals of Probability, Continuous Probability Distributions, Multivariate Probability Distributions, and Stochastic Processes and Time Series, let's explore a use case related to stock price prediction using Autoregressive Integrated Moving Average (ARIMA) models. This use case involves applying time series analysis techniques to forecast future stock prices.

Description:

In this use case, we will focus on modeling and predicting the future stock prices of a publicly traded company using historical stock price data. We will employ ARIMA models, a popular tool in time series analysis, to capture the underlying patterns and trends in stock prices and make short-term predictions.

Key Components:

Fundamentals of Probability: Probability concepts, such as sample spaces and events, provide a foundational understanding of randomness, which is inherent in stock price movements.

Continuous Probability Distributions: Knowledge of continuous probability distributions, including the normal distribution, is valuable for modeling the statistical properties of stock returns.

Multivariate Probability Distributions: Concepts related to joint probability distributions and covariance are relevant when analyzing multiple stocks or asset classes.

Stochastic Processes and Time Series: Understanding stochastic processes and time series analysis, particularly ARIMA models and Brownian motion, is crucial for modeling and predicting stock prices.

Python Code Example (Stock Price Prediction with ARIMA):

```python
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.arima_model import ARIMA

# Load historical stock price data
data = pd.read_csv('stock_price_data.csv', parse_dates=['Date'],
index_col='Date')

# Extract the closing price as a time series
stock_price = data['ClosingPrice']

# Fit an ARIMA model
model = ARIMA(stock_price, order=(2, 1, 1))  # Example order (p, d, q)
model_fit = model.fit(disp=0)

# Forecast future stock prices
forecast_steps = 10
forecast, stderr, conf_int = model_fit.forecast(steps=forecast_steps)

# Plot historical and forecasted stock prices
plt.figure(figsize=(10, 6))
plt.plot(stock_price, label='Historical Price', color='blue')
plt.plot(pd.date_range(start=stock_price.index[-1],
periods=forecast_steps+1, closed='right'), [stock_price.iloc[-1]] +
list(forecast), label='Forecasted Price', color='red', linestyle='--')
plt.xlabel('Date')
plt.ylabel('Stock Price')
plt.title('Stock Price Prediction with ARIMA Model')
plt.legend()
plt.grid(True)
plt.show()
```

In this code, we load historical stock price data, fit an ARIMA model to the time series, and use the model to forecast future stock prices. The ARIMA model captures the autocorrelation and differencing patterns in the data, allowing us to make short-term predictions.

This use case demonstrates how probability and time series analysis techniques can be applied to predict stock prices, assisting investors and financial analysts in making informed decisions.

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TEXTE DE DESCRIPTION DU COURS

In the realm of stochastic dynamics and probability, there exists a course that takes you on an expansive journey through chance, uncertainty, and the innate patterns underlying the chaos. This course, while rooted in the rigorous realm of mathematics, unfurls like an intricate tapestry of stories and discoveries.

You start your expedition by grounding yourself in the essentials. Here, you decode the mathematical dialects in which probability whispers its truths, understanding the vast universes of potential outcomes and the events that punctuate them. As you wander deeper into this foundational realm, you encounter the enigmatic nature of conditional probabilities. Here, cause and effect play a delicate game, as events either dance in synchronicity or remain fiercely independent. And just when you think you're grasping the rules of this dance, the profound insights of Bayes' Theorem emerge from the shadows, offering revelations about past events based on new information.

Leaving the world of the discrete behind, you venture into the undulating landscapes of continuous probability distributions. These are lands shaped by forces like the uniform spread of raindrops or the exponential decay of radioactive substances. But among these terrains, one stands majestic and all-encompassing: the bell curve of the Normal distribution, a symbol of the inherent predictability in a world of randomness. Journeying through this landscape, you'll unearth gems like the Central Limit Theorem, a testament to the omnipresent nature of the Normal distribution, even when forged from non-normal origins.

Your travels then guide you into the multi-dimensional labyrinths of multivariate distributions. Here, the relationships between variables play out in grand dramas, revealing stories of covariance, tales of correlation, and the nuanced choreography of variables in the multinomial and multivariate arenas. But just as the paths seem familiar, you're led into an unexpected detour: the esoteric world of Copulas. These mathematical functions, with their ethereal beauty, weave together the individual stories of variables, crafting tales of interdependence that find their echoes in real-world arenas like finance and risk analysis.

Emerging from this journey, you're no longer just a student of stochastic dynamics and probability; you're a storyteller, a mathematician, and an explorer, equipped with the knowledge and wonder of the vast probabilistic landscapes you've traversed.